Chapter 1

Understanding school mathematics

Big Ideas

- Sophisticated mathematical skills are inherent in many daily activities.
- School mathematics needs to change and is changing.
- Learning mathematics is both an individual and social enterprise.
- Teaching mathematics is a rewarding but complex and demanding task.

Chapter objectives

This chapter will enable the reader to:

- Understand the contribution of mathematics to society and the role of school mathematics in a futures-oriented curriculum
- Develop a broad understanding of what is involved in the teaching and learning of mathematics in contemporary Foundation to Year 9 classrooms
- Appreciate the range of issues and challenges impacting on the provision of school mathematics at this level
- Recognise the important role of reflection and research in mathematics education.

Key terms

Communication patterns
Conceptual understanding
Curriculum
Knowledge for teaching mathematics
Mathematical problem solving
Mathematical reasoning
Open-ended questions
Pedagogical content knowledge
Pedagogy
Problem solving
Procedural fluency
Representations
Rich tasks

Everyone can do maths

'I was never any good at maths ... I dropped it as soon as I could.' Most teachers of mathematics experience this reaction in a social context when they are asked their occupation. Such responses are amazing when you consider that the same people would probably not have made the same claims about their capacity to read or write. It seems that it is socially acceptable to admit to disliking or not being 'very good at' mathematics. It is a sad irony that those who profess such views frequently demonstrate sophisticated uses of mathematics in their everyday activities. For instance, I recall a taxi driver who, having confessed that he had 'failed mathematics in Year 9', expertly gauged the flow of the traffic, consulted a global positioning system, decided to take an alternative route to ensure we arrived in time, and at the end of the journey mentally added the airport tax to the fare and calculated the change.

1. When was the last time you truly, madly, deeply, really enjoyed doing some mathematics? What did you do and how did it make you feel?
2. Can you recall a teacher who made a significant impact on your learning of mathematics or an experience when you were 'turned off' mathematics? If so, describe the circumstances in terms of who, what, when and where. How did it make you feel?
3. What lessons can be learnt from this?

Introduction

We introduced this book by talking about teacher quality and what it means to be an effective teacher of mathematics. While it is relatively easy to list generic characteristics, what matters is what students experience, individually and collectively. Teacher quality is clearly related to teacher knowledge and confidence, but what actually happens in classrooms is also affected by what teachers feel they have to teach, how they go about teaching it, and the social contexts in which teaching takes place.

Teachers choose to teach for a variety of reasons—and it is rarely for the money! The most commonly cited reason for choosing to teach is 'to make a difference'. In *A Sense of Calling: Who teaches and why*, Farkas, Johnson and Foleno (2000) report that 96 per cent of new teachers surveyed reported that they chose teaching because it 'involves work they love doing' and gave them 'a sense of contributing to society and helping others' (p. 11). 'Understanding what matters to people, what motivates them and why they do what they do can make the difference between a conversation that moves forward and one that goes nowhere' (p. 8).

Given that most teachers are motivated by a desire to support the learning and well-being of others, and given what we know about effective mathematics teaching, why is it that students' experiences of learning mathematics are not that much different from what they were in the past, particularly in the middle years of schooling (Bodin & Capponi, 1996; Yates, 2005)? It appears that teachers tend to teach mathematics in the way that they were taught (e.g. Brady, 2007; Stigler & Hiebert, 1999) and that their decisions about content and pedagogy are very much related to what they know and believe about the nature of mathematics and the teaching and learning of mathematics (e.g. Brady, 2007; Fennema, Carpenter & Peterson, 1989; Handal & Herrington, 2003, Siemon, 1989).

Consider and discuss your teaching

JOB, VOCATION OR CAREER?

For me, in Year 12, teaching was the obvious choice. Where else do you get an opportunity to do all the things you like doing, such as learning mathematics, studying history, discussing literature, playing sport, singing and performing. But most of all, I became a teacher because I somehow felt it was the right thing to do. Helping others learn is not only very satisfying, it increases people's opportunities in life. Education is the key to making the world a better, fairer, more environmentally responsible place.

1 What were your reasons for choosing to become a teacher? Was it because of particular strengths and interests, role models, or opportunities?
2 What do you hope to achieve as a teacher of mathematics?
3 What do you see yourself doing in ten years' time?

In what follows we will look at what we mean by mathematics, the goals of contemporary school mathematics, and the influences, including teacher knowledge and beliefs, that act to promote (or constrain) the teaching and learning of mathematics in the primary and middle years of schooling.

What is mathematics?

We all have views on what constitutes mathematics, and these views shape our decisions about how we teach and learn mathematics. For instance, if mathematics is viewed as a set of universal truths, teachers are more likely to see their task as transferring a given set of facts and skills to students and to view student learning as the capacity to reproduce these facts and skills as instructed. If, on the other hand, mathematics is viewed as a socio-cultural practice, a product of reflective human activity, then it is more likely that teachers will see their task as engaging students in meaningful mathematical practices and view student learning in terms of conceptual change.

Many views have been advanced to describe the nature of mathematics:

> Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. (Courant & Robbins, 1941, p. xv)

> Mathematics reveals hidden patterns that help us understand the world around us … Mathematics is a science of pattern and order. (Mathematical Sciences Education Board, 1989, p. 31)

> [A]ll cultural traditions … find connecting patterns and apparent symmetries throughout nature. Individually and collectively, we accept these patterns as part of the background of our lives: beyond question or doubt, all Australians agree we live in an orderly, knowable universe. Yet the patterns which organise, and the laws which govern, European knowledge and perception apparently have little in common with the patterns which make sense of the Aboriginal world … both kinds of patterning form complex and mathematically sophisticated systems, both have powerful ideological underpinnings, both are social constructions emerging from historically identifiable contexts, both attempt to account for natural as well as social phenomena, both are rational in principle and practical in application. (Watson, 1989, p. 31)

These views all propose that mathematics offers a way of understanding the world we live in. Mathematics provides a consistent framework, a symbolic technology, by which we can model 'reality', solve problems, and support predictions, but how this is described and communicated depends very much on our cultural values and traditions. For instance, our taken-for-granted view of mathematics—the mathematics of our schooling that Bishop (1991) refers to as 'Mathematics with a capital "M"'—is governed by number patterns and relations. There are other cultures whose ways of understanding the world are not governed by 'how much and how many' but by complex kinship patterns that connect all things to each other without the need of numbers (e.g. Watson, 1989; Bishop, 1991). In other words, mathematics is a 'pan-cultural phenomenon: that is, it exists in all cultures', and 'Mathematics' is a 'particular variant of mathematics, developed through the ages by various societies' (Bishop, 1991, p. 19).

This is a difficult notion to grapple with, as we can scarcely imagine a world without Mathematics, and yet we happily accept that different languages are developed by different cultural groups in order to communicate. This prompted Bishop to ask: if language develops from the need for, and activity of, *communicating*, what are 'the activities and processes which lead to the development of mathematics?' (1991, p. 22). According to Bishop, there are six fundamental mathematical activities that occur in some form across all cultures, which he describes as follows.

- *Counting*—the use of a systematic way to compare and order discrete phenomena. It may involve tallying, using objects or string to record, or special number words or names
- *Locating*—exploring one's spatial environment and conceptualising and symbolising that environment, with models, diagrams, drawings, words or other means
- *Measuring*—quantifying qualities for the purposes of comparison and ordering, using objects or tokens as measuring devices with associated units or 'measure-words'
- *Designing*—creating a shape or design for an object or for any part of one's spatial environment. It may involve making the object as a 'mental template', or symbolising it in some conventional way
- *Playing*—devising, and engaging in, games and pastimes, with more or less formalised rules of play that all players abide by
- *Explaining*—finding ways to account for the existence of phenomena, be they religious, animistic or scientific (Bishop, 1988, pp. 182–3).

In looking at these activities and processes, we can see how it might be possible for there to be many mathematics underpinned by different cultural norms and values. For Bishop, mathematics can be viewed as a 'way of knowing'. The cultured nature of mathematics is illustrated by an apocryphal story that describes a group of tourists who were being escorted through the central Australian desert by a local Indigenous man. When they had been travelling for some time and were feeling overwhelmed by the seemingly unchangeable landscape, one of the tourists asked the guide, 'What do you do when you get lost out here?' The man turned to the tourist somewhat bemused, and said, 'I go home.' In other words, for him, there was no notion of 'lost'. He knew the landscape intimately, and he could discern its patterns and variations. By this means he knew exactly where he was and in which direction he needed to travel to return home, without the need of GPS, compass or paper map.

Mathematics as we know it evolved from the earliest civilisations of Egypt, Greece, Mesopotamia, India and China. These were established in river valleys where there was plenty of water, fertile soil, and a climate conducive to an agrarian economy. It is not too difficult to imagine a scenario where competition for land and resources to sustain rapidly growing populations led to a system of quantifying to settle disputes and support trade. Land could be identified by precise measures, permanent structures could be erected, goods could be exchanged on a comparable basis, taxes could be levied on property and produce, time could be measured by reference to the sun and the stars, and journeys could be described in terms of units of length, time and direction. When trade and travel expanded, systems of quantifying were used to transcend language barriers.

Mathematics was also pursued as an intellectual pursuit in its own right. For example, the Pythagoreans noticed that for right-angled triangles, the sum of the squares on the smaller sides was equal to the square on the longest side, the hypotenuse. At the time, the only numbers that were recognised were the natural numbers (1, 2, 3, 4, …) and numbers that were ratios of these numbers (e.g. 1:2, $\frac{4}{2}$, or $\frac{17}{3}$). Subsequent explorations of right-angled triangles in which the two smaller sides were both one unit in length led to the proposition of a new type of number, in this case the square root of 2. Although resisted at first, this proposition eventually led to the recognition of the irrational numbers, which are numbers that cannot be expressed as ratios of natural numbers. The irrational numbers include pi (π) and non-terminating decimals. This example illustrates an interesting philosophical question: were the irrational numbers always there, 'waiting to be discovered', or were they the product of human reflective activity at a certain time in a certain cultural setting? This question is often characterised as a debate between *absolutist* and *fallibilist* views of mathematics—that is, between mathematics as a set of irrefutable truths, and mathematics as a socially constructed practice (e.g. Bishop, 1991; Ernest, 1991; Presmeg, 2007). While these views may co-exist to some extent, what teachers believe about the fundamental nature of mathematics has important implications for practice as we shall see below.

Goals of school mathematics

Niss (1996) suggests that there are three fundamental reasons for the existence of school mathematics as we know it today. They are that the study of mathematics contributes to:

> the technological and socio-economic development of society; the political, ideological and cultural maintenance and development of society; [and] the provision of individuals with prerequisites which may assist them in coping with private and social life, whether in education, occupation, or as citizens. (p. 22)

In ancient times, the study of mathematics was restricted to a select few, generally young men with the time to converse with more learned others. For some, this was simply an enjoyable pastime, an intellectual pursuit for its own sake, while for others such as the Pythagoreans, in Greece, this activity was shrouded in secrecy and mysticism. However, as production became more specialised and trade expanded, mathematical know-how was acquired as a matter of necessity. Merchants, builders, navigators, tax collectors, artisans and religious leaders all needed some mathematics, and they inducted those that followed them into their particular mathematical practices.

Before the nineteenth century, there was little offered in the way of formal education beyond the opportunity to learn a trade. For the very few who did receive some sort of formal education, this was largely justified on the grounds that it contributed to 'the political, ideological and cultural maintenance of society' (Niss, 1996, p. 22). This situation changed with the advent of the industrial age and the introduction of compulsory schooling. Elementary mathematics was included as a core component of the curriculum, presumably 'to contribute to the technological and socio-economic development of society, while at the same time placing some emphasis on equipping individuals with tools for mastering their vocational and everyday private lives' (Niss, 1996, p. 23).

Consider and discuss your maths

FROM AN 1875 BOOK OF ARITHMETICAL EXAMPLES

> A wine merchant buys 3 hhds. of wine at Bordeaux, at £15 per hhd., pays duty 1 s. per gallon, and carriage £3. What must he sell the wine at per gallon, to clear £10 in the transaction? (Davis, 1875, p. 47)

1 What does this say about the assumptions underpinning the teaching and learning of mathematics at the time? How different are these from the assumptions underpinning your experience of school mathematics? Discuss in terms of the three fundamental reasons for studying mathematics described by Niss above.
2 What knowledge and skills are needed to solve this problem? (Hint: you may need to rewrite this in a more familiar form first.)

In the same book, a 'simple multiplication' problem was given as 856 439 082 × 7 008 001 (p. 13). These days, it is difficult to understand why this was regarded as 'simple' or why such a problem would be set but these examples nicely illustrate that school mathematics is a human construction reflecting societal values and priorities at particular points in time. The fact that all students, not just a select few, were expected to solve these problems is indicative of the goals of school mathematics at the time and the assumptions made about the teaching and learning of mathematics. Contrast this with how an Aboriginal child living in a remote part of Australia in 1875 might have been expected to learn to find his or her way home in the desert.

A shift in emphasis

Today, all three of Niss's reasons for including mathematics in school curricula can be found in contemporary curriculum documents, for example, the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000) and the

Australian Mathematics Curriculum (Australian Curriculum Assessment & Reporting Authority [ACARA], 2015), which aims to ensure that students:

> are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens … develop an increasingly sophisticated understanding of mathematical concepts and fluency with processes, able to pose and solve problems and reason in number and algebra; measurement and geometry; and statistics and probability … [and] recognise connections between the areas of mathematics and other disciplines and appreciate mathematics as an accessible and enjoyable discipline to study. (p. 1)

Think + link: The chapters in Part 3 examine what we mean by numeracy and explore its relationship to school mathematics in Years F–9. In particular, these chapters focus on the 'big ideas' in mathematics and how these can be applied in everyday contexts to solve problems and inform decision making.

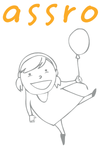
It is interesting to note the order in which these aims are stated. The first echoes Niss's (1996) third reason for studying mathematics, that is, the 'provision of individuals with prerequisites which may assist them in coping with private and social life, whether in education, occupation, or as citizens' (p. 22). This reason for studying mathematics has become much more prominent recently because of its association with *numeracy*, which is defined as the 'effective use of mathematics to meet the general demands of life at school and at home, in paid work, and for participation in community and civic life' (National Numeracy Benchmarking Taskforce, 1997, p. 30).

This shift in emphasis arose in response to concerns about the capacity of individuals to function effectively in a modern technological society, in particular to critically question public policy decisions and understand the ways in which mathematics is being used to monitor and shape our lives (Gellert & Jablonka, 2008).

We have never needed mathematics more

While it is true that technology has replaced many of the routine procedures traditionally taught in school mathematics, the advent of sophisticated information and communication technologies has also drastically changed the way we conduct our everyday lives. Mathematics underpins much of this technology, and mathematics is used in increasingly powerful and subtle ways to persuade voters and consumers to act in certain ways. This suggests that we have never needed mathematics more. But the mathematics we need is not the school mathematics of the past with its emphasis on isolated skills and rote learning.

For the classroom



ACTIVITY 1.1 Images of mathematics in use

Collect a range of newspapers, magazines, food containers and screen dumps from a variety of websites. Ask students to identify where mathematics is being used and how. Encourage younger students to look beyond numbers to measures, and older students to look for instances of proportion and other, more subtle, uses of mathematics, such as location of visual displays and headings. Represent as a collage of annotated images for classroom display.

Today, individuals need to be able to make sense of vast amounts of quantitative and spatial information presented in increasingly sophisticated multimedia formats; make decisions on the basis of that understanding; and communicate their reasons for doing so if challenged. According to Becker and Selter (1996), the 'ultimate objective of student learning at all levels is the acquisition of a *mathematical disposition* rather than the absorption of a set of isolated concepts or skills' (p. 542). In their opinion, students should learn to:

> *be creative*: to look for patterns, make conjectures, generate new problems … to *reason*: to give arguments, uncover contradictions, distinguish between facts and assertions … to *mathematize*: to collect data, process information, interpret data and solutions … and to *communicate*: to express their own thoughts, accept the ideas of others, establish forms of cooperation. (p. 542)

This is consistent with Bishop's view of mathematics as 'a way of knowing' rather than 'a way of doing'. The values they represent are reflected in the *Australian Curriculum: Mathematics* (ACARA, 2015) in the form of proficiencies, specifically:

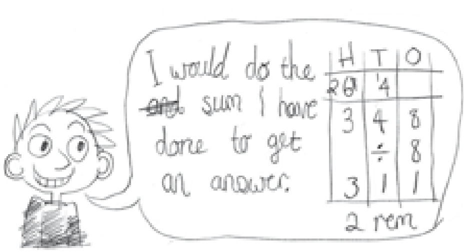
- conceptual understanding
- procedural fluency
- problem solving (or strategic competence), and
- adaptive reasoning.

Think + link:
Chapter 4 considers what is involved in building mathematical proficiency and the confidence and competence to view the world mathematically.

From the student's perspective

Student responses to mathematical tasks can reveal a lot about their perceptions of the goals of school mathematics. Nick was a Year 4 student who knew his number facts and was 'good at' 'doing division' by the traditional 'goes into' [sometimes referred to as 'guzinta'] algorithm. For example, he could confidently divide 364 by 7 by saying '7 goes into 3? No, carry the 3. 7 goes into 36 … yes, 5 times and 1 left over [writes 5 underneath and 1 beside the 4], 7 goes into 14, yes 2 times [writes 2]'. After some time in class exploring an alternative approach involving sharing and base 10 materials, which Nick clearly understood and could justify to his peers, he was given the problem 'Eight families shared a prize of $348. How much did each family receive?' Nick provided the following 'talking head' response. He drew a face to show how he felt about the task and wrote a brief explanation of what he had done.

Figure 1.1 Nick's response to a division problem



Source: Siemon, 1993

Think + link: Chapter 2 examines the role of beliefs and values in learning school mathematics and stresses the importance of conceptual understanding as a basis for procedural fluency.

Despite his demonstrated understanding of both forms of division, Nick chose to create his own algorithm based on what he knew about subtraction and renaming. In this case, reading from the top down and starting with the ones he reasoned, '8 how many 8s? … 1 … 4 how many 8s? Can't do, so trade a hundred, 14 tens how many 8s? … 1 and 6 over', which he records over the 2 in the hundreds place. Realising '2 how many 8s?' is not going to work, he crosses the 6 out and rewrites the 2 and the 6 as 26 and proceeds, '26 how many 8s? … 3 and 2 remainder', which he records. His comments indicate his beliefs about what he believes school mathematics is about, that is, using 'sums' to get answers. When asked about his answer, Nick, said, 'Oh if it was real money I wouldn't do it like that.' Prompted to explain how he would do it, Nick replied, 'Well 8 families, $40 each that'd be $320, $50 each would be $400, I reckon it's about $43.' Nick's problem was not with division, but with the values and beliefs he held about the nature and purpose of school mathematics. Asked why he did this, Nick said that he knew his 'old way of doing it would work but Mrs … didn't like that' and he could do it the new way 'but that was too long'.

Affordances and constraints

Teaching mathematics is a complex, demanding, but rewarding task that requires a knowledge of students, of content, and of how that content might best be represented to engage and support learners. Referred to by Shulman (1986) as **pedagogical content knowledge** (PCK) and, more recently and with respect to mathematics teaching and learning, as **knowledge for teaching mathematics** or KTM (Ball, Thames & Phelps, 2008), it clearly involves much more than a knowledge of mathematics at the level taught.

> Teaching is about building relationships—between students and the teacher and among students themselves around mathematics—and engaging together in constructing mathematical meaning. Teaching involves orchestrating the content, the representation, and the people in relation to one another. It is about making decisions in the moment that serve the individuals and collective. It is about understanding the students … It is about working together to negotiate meaning. (Franke, Kazemi & Battey, 2007, p. 228)

Think + link: The characteristics of effective teachers are discussed in more detail in Chapter 3 and in Parts 4 and 5 of this book

Teachers need to have a deep understanding of what makes particular mathematics content difficult to learn (content knowledge), what representations and instructional strategies are best suited to the needs of individual students (pedagogy), and how to manage the relationships in which the teaching and learning takes place (social context). Although indistinguishable in practice, these three key aspects of teacher knowledge are briefly described below.

Knowledge of content

In Australia, decisions about which mathematics to teach are largely guided by the *Australian Curriculum: Mathematics* (ACARA, 2015) and supplementary support material developed by state and territory education authorities. The Curriculum is described by Year level (F–12) across three strands and four proficiencies, as shown in Table 1.1.

Table 1.1 Structure of the *Australian Curriculum: Mathematics*

CONTENT STRAND	PROFICIENCIES
Number and Algebra	[Conceptual] Understanding
Measurement and Geometry	[Procedural] Fluency
Statistics and Probability	**Mathematical Problem Solving**
	Mathematical Reasoning

Source: ACARA, 2015

A major shift in the curriculum is the recognition of the importance of *big ideas*. A 'big idea' provides an organising framework that encompasses and connects a number of related ideas and strategies and supports further learning and generalisations. For example, *multiplicative thinking* not only encompasses the various meanings and representations of multiplication and division, together with a range of appropriate solution strategies, but also supports connections between these operations and the base 10 system of numeration, the rational numbers, and generalisations associated with proportional reasoning.

Modern curriculum also attempts to build on what is known about the progression of children's thinking towards the big ideas. Referred to as *learning trajectories* (Simon, 1995), teachers need to be aware of the developmental pathways, intermediate goals, and the instructional strategies, tasks, and representations by which student thinking is likely to be further enhanced.

Think + link: The 'Consider and discuss' and 'Teaching challenges' sections throughout this book are designed to assist you to build your mathematical knowledge for teaching.

The language of mathematics

Building a vocabulary

Teachers need a vocabulary to reflect on their practice, share their thinking and decision making, and engage in further professional learning. In the chapters that follow we will endeavour to introduce the terms commonly used in relation to the teaching and learning of mathematics in a clear and consistent manner. A glossary of key terms is provided at the end of this book.

The many faces of curriculum

Curriculum comes in many different forms. The curriculum produced by education systems is often referred to as the intended curriculum. It sets the standard for what is valued and what will be assessed at a system or national level. However, schools and teachers need to interpret the intended curriculum in the light of their own knowledge and experience and what they know about their particular student population. This can lead to subtle and not-so-subtle variations in the curriculum that are actually translated into practice and assessed. These versions of the curriculum are often referred to as the implemented curriculum and the evaluated curriculum respectively.

These variations highlight the competing tensions faced by teachers as they try to balance the learning needs of the students with system and community expectations. This is a non-trivial task that can lead to underachievement or learners being left behind if it is not managed well. High but realistic expectations of student learning are associated with effective mathematics teaching (e.g. Hattie, 2003). This requires accurate information about what each student knows and is able to do, a deep knowledge of the key ideas and strategies needed to progress student learning, and a sound knowledge of how this can be achieved.

Pedagogy

Although originally used to refer to the 'art and science' of teaching children, **pedagogy** is now more broadly understood as the 'knowledge and principles of teaching and learning' (Griffith & Kowalski, 2010, p. 111), the art and profession of teaching or, more particularly, instructional strategies or a style of instruction.

Research on teaching and learning and developments in technology have prompted considerable changes in how mathematics is taught. For instance, it is now recognised that:

> Learning mathematics is basically a constructive process … pupils gather, discover, create mathematical knowledge and skills mainly in the course of some social activity that has a purpose … Instead of being the main if not the only source of information, the teacher becomes a 'privileged' member of the knowledge building community of the classroom who creates an intellectually stimulating climate, models learning and problem-solving activities, asks provocative questions, provides support to students through coaching and guidance, and fosters students' responsibility for their own learning. (Verschaffel & De Corte, 1996, p. 102)

Think + link:
Chapters 2 and 3 consider what we know from research and practice about the learning and teaching of mathematics in Years F–9.

As a consequence, school mathematics now emphasises interaction, collaboration and a variety of organisational styles (e.g. Boaler, 2002). Teachers are encouraged to focus on important mathematics through the use of challenging problems, extended investigations, **rich tasks**, **open-ended questions**, games, mental computation, the discussion of solution strategies, visualisation, and the appropriate use of materials and representations supported by appropriate information and communications technologies. In short, quality mathematics instruction involves a rich and varied supportive learning environment in which all learners feel they have a place and the capacity and desire to contribute to the collective enterprise.

We now know a lot more about how mathematics is learnt. Meaningless, mind-numbing, text-based drill and practice, and doing it one way, the teacher's way, does not work. Concepts need to be experienced, strategies need to be scaffolded, and everything needs to be discussed.

Think + link:
Chapter 5 examines communication in more detail, and Chapter 8 considers the challenges and affordances of mixed ability, diverse classrooms.

Communication

From the student's perspective, the critical element in their learning is the quality of teacher explanations, in particular the capacity of teachers to connect with their level of understanding and communicate effectively. As Vincent, a Year 9 student identified as 'at risk' by his teachers, said so eloquently, 'change the way it's explained, they need to think about how you understand, not how they explain' (Siemon & Virgona, 2001, p. 49).

Teachers not only need to know the key concepts, skills and strategies that underpin the mathematics they are teaching, but also need a deep knowledge of the links between these ideas, what makes them difficult, and how they are best communicated. **Communication patterns** in mathematics classrooms are shaped by a number of factors, including the established or assumed social norms of the classroom, students' cognitive, cultural and/or linguistic background, and the beliefs and attitudes of both teachers and students. In the past, teachers typically asked relatively low-level, closed questions, students responded, and the teacher evaluated the response in some way. Known as the initiation-response-evaluation, or IRE, form of interaction, this is now regarded as fairly restricted in comparison to the more productive forms of interaction that have been recognised in recent years (e.g. Wood, 1994; Siemon, Cathcart, Lasso, Parsons & Virgona, 2004).

Representations

Teachers draw on a range of materials (also referred to as manipulatives), models, and **representations** to support communication in mathematics classrooms. While these are valuable as tools to support the negotiation of meaning, they do not, and cannot, convey meaning. Indeed, many, such as fraction diagrams, presuppose the learning they are intended to support. Language plays an important role in mediating the use of such tools and negotiating learning. Teachers need to be critically aware of the assumptions underpinning the use of materials and representations, as well as of the language and questioning needed to scaffold their use. If these materials and representations are seen purely as an interesting game or as the only means of finding a solution, then their use needs to be questioned.

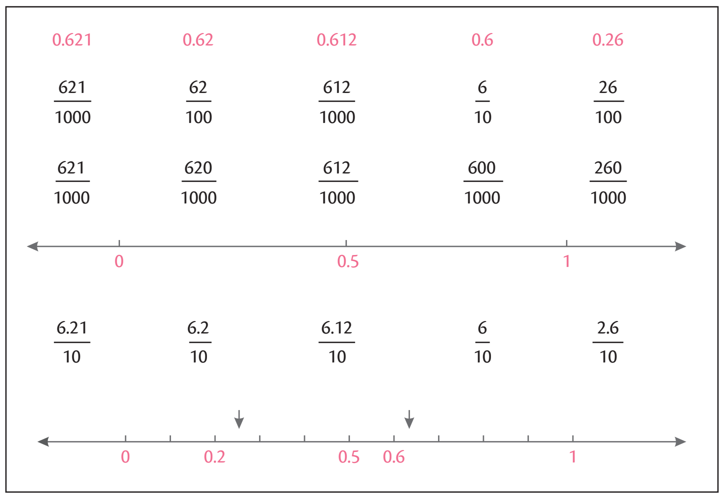
Sometimes our taken-for-granted models can prevent us from 'seeing' or 'hearing' what it is that students are trying to tell us. For example, many years ago, about 10% of our first year teacher education students ordered a set of decimal fractions from smallest to largest as follows: 0.621, 0.62, 0.612, 0.6 and 0.26. Initially, I thought this was simply an oversight but when the same response was observed the following year, I invited the students to a meeting to explore their reasoning. The following is a slightly abbreviated version of the discussion that took place. (L stands for myself as the lecturer, S1, S2, etc stands for different students.)

> L: Can you tell me why you did it this way? [pointing to the question and response recorded on the whiteboard]
>
> S1: Because it asked for smallest to largest. [others nod in agreement]
>
> L: So, this is the smallest? [pointing to 0.621 on the left]
>
> S2: Because, it's the furthest away from zero. [again, others nod in agreement]
>
> L: That's interesting. I agree it's the furthest away from zero but I think it's the largest for the same reason.

At this point, I suggested we rename the decimal fractions as common fractions and I asked, 'Do you still think these are ordered from smallest to largest?'. They responded fairly quickly: 'Oh, you can't tell. The denominator needs to be the same.' At that point we proceeded to rename the common fractions as thousandths (see Figure 1.2). However, when I repeated the question about the order of the common fractions, pointing to the row of thousandths, the response was surprising: ‘Oh, when it’s like that, it’s the other way round.’ This was met with nods of agreement and the following discussion ensued.

> L: Does that mean that your answer up here [pointing to the decimal fractions at the top of the whiteboard] … the order … can that be reversed?
>
> s3: No. When it’s like that [referring to the decimal fractions], that’s right.
>
> L: But don’t you agree we haven’t changed anything here; we’ve just renamed the decimal fractions as common fractions? [long period of silence]
>
> S1: I’m not sure. It’s just the way it is.

Figure 1.2 Teaching response to decimal misconception



Source: Siemon (1999)

Perplexed by this response, I drew a number line on the board, marked the points for 0 and 1 and asked the students to identify where 0.5 should be located. Most agreed that it should be placed half-way between 0 and 1. Having done that, I asked if this gave a clue as to where any of the decimal fractions might live, for example 0.6. With some uncertainty, it was agreed that 0.6 lived just a bit further to the right of 0.5, at which point I sketched in the remaining tenths. When it became clear that the students were uncertain where any of the other decimal fractions lived, I suggested we rename them all as tenths. I then used arrows to demonstrate that 6.21 tenths was a bit more than 6 tenths and 2.6 tenths was more than halfway between 2 tenths and 3 tenths. Although it was clear that I had lost them, I nevertheless asked:

> L: Does that help? [very long pause]
>
> S2: Sort of … [another very long pause] … you know, you said … it goes thousands, hundreds, tens, ones, tenths and hundredths …
>
> L: Yes, the place-value parts [writing these on the board as Th H T O t h]
>
> S2: Well, … [coming out to the board and standing for a while in front of the 0 on the number line] … if the ones, tens, hundreds, thousands and so on live down here [walking to the

right and beyond 1 on the number line] … then [returning to 0 and walking in the opposite direction] … the tenths, hundredths, thousandths live down here.

L: Oh, now I see why you think 0.621 is the smallest, you think it lives down here. [pointing to the left of 0 on the number line; vigorous nods from at least four students]

S1: Yes, that's how I do it … I never thought about it before, I just did it that way.

This indicates that a small but significant number of these primary pre-service teachers had independently constructed a shared view of the number line that was completely at odds with the conventional view. It is hard to conceive that they were ever taught or exposed to this view, so where has it come from? I suspect that it was the apparent linear ordering of the place-value positions that motivated their thinking, together with a view of the number line as a count of discrete chunks rather than an infinite set of points.

I asked them how they managed to pass Year 11 maths, and the response was 'we learnt it off by heart'. It is hardly surprising that their mathematics teachers did not understand—they would be operating, as indeed I was, on the basis of an assumed model of the number line and would never have imagined that this other construction might exist. Indeed, it was entirely fortuitous that this conception came to light at all.

From this experience I learnt never to underestimate the capacity of students to construct their own mathematical meanings—what might pass for learning on the surface may well be founded on misconceptions It is wise to probe student thinking continuously and interrogate one's own assumptions and understanding of mathematics on a regular basis.

Think + link: Chapter 2 discusses the importance of prior learning and understanding in the construction of shared mathematical meaning, and Chapter 6 examines the role of materials and representations in this process.

Think + link: Chapter 7 discusses techniques for probing student understanding in more detail; in particular, it will describe the talking heads approach illustrated above to elicit Nick's strategies and beliefs.

Meaningful contexts

In 1985 Carraher, Carraher and Schliemann reported a study that involved school-aged children working as street vendors in Reciffe, Brazil. They had observed how efficiently the children had calculated the price of goods in the marketplace, and were interested to see if they could solve similar problems in a school context. The researchers identified the mathematics involved in the marketplace transactions, and created a comparable, context-free, symbolic problem and a related word problem. When these were presented in a school environment, the children attempted to use school-taught procedures but were largely unsuccessful although they were more likely to solve the related word problem correctly than the symbolic one. At the time, this research was used as an argument to include 'real-world' mathematics and multiple strategies in school pedagogy.

CHILDREN'S STRATEGIES

Consider and discuss your teaching

1 What are the implications of this research for the teaching and learning of school mathematics?
2 How might the students have learnt to carry out these calculations? Are they likely to work in all situations?
3 There is an argument that these responses say more about the students' beliefs and expectations of schooling than about their capacity to solve problems. Do you agree?

For the classroom

ACTIVITY 1.2 **Where we use mathematics**

- Invite students to make a video or group poster of the contexts in which they or their friends use mathematics on a reasonably regular basis outside school.
- Compare and discuss examples and use the contexts to create a bank of similar problems.

Think + link: Chapter 4 discusses problem solving and the importance of using real-world contexts, in considerably more detail.

Problem solving is a one of the most important reasons for studying mathematics. But, as indicated above, the problems have changed significantly from the days of our great-great-grandparents. Today's problems can be found in popular culture more than in everyday calculations, and we need to be able to reason mathematically in order to interrogate the claims and counter-claims that are made in increasingly sophisticated ways. For example, mobile phone companies advertise their phone plans in ways that make it very difficult to decide which plan is the most appropriate. Solutions to these types of problems require a capacity to model and investigate complex situations, seek and process data, and communicate this effectively in a much wider range of ways than was the case in the past.

Assessment

Assessment plays a critical role in teaching and learning. There are two main purposes of assessment. The first, sometimes referred to as assessment of learning, is used for reporting purposes; it compares achievement against expected norms or standards. This information can be used to inform school-based decisions about curriculum offerings (e.g. the amount of time spent on geometry), education policy more generally, and system-wide initiatives such as additional professional development or targeted resourcing (e.g. Rowe, 2006). However, while this form of assessment is useful, it cannot be used to inform teachers about where to start teaching with individual students.

Think + link: Chapter 7 explores the purposes and forms of assessment in mathematics and discusses the important role of feedback in student learning.

Scaffolding student learning is the primary task of teachers of mathematics, but this cannot be achieved without accurate information about what each student knows already and what might be within their grasp with some support from their teacher and/or peers. This second form of assessment, *assessment for learning*, requires assessment tasks and techniques that expose students' thinking in ways that can be interpreted in terms of a recognised developmental progression of related ideas and strategies, ideally an evidence-based learning trajectory such as the *Learning Assessment Framework for Multiplicative Thinking* (Siemon, Breed, Dole, Izard & Virgona, 2006a).

Social context

Teachers' decisions about what they do in classrooms are powerfully shaped by what they know and believe about the nature of mathematics and the teaching and learning of mathematics. They are also shaped by the behaviour of their students and the social context in which the teaching of mathematics is situated (Fennema et al., 1989). Poverty, racism, isolation, language background and physical disability are among the many factors known to affect education outcomes. While schools alone cannot redress these inequities, there is much that schools and teachers can do to improve the education outcomes of students from disadvantaged backgrounds. For instance, we know from middle years research (e.g. Siemon & Virgona, 2001; Siemon et al., 2006) that the quality of teacher explanations, choice of instructional strategies, and capacity to recognise where to start teaching are crucial in meeting the needs of at-risk learners.

Think + link: Chapter 8 explores what we mean by diversity and the implications of this for the teaching and learning of mathematics.

The social and physical arrangements of classrooms have changed since the days when it was assumed that the purpose of education was the transfer of knowledge into the 'empty heads' of children. Classrooms are now seen as collaborative communities of inquiry, with accepted ways of interacting (social norms), and negotiated ideas about what constitutes an explanation or what counts as a different solution, an efficient strategy, or a useful representation, all of which are examples of socio-mathematical norms (Yackel & Cobb, 1996).

The issues of agency (available means) and identity are often ignored in programs designed to improve school mathematics outcomes or attempts to understand why some students are indifferent or resistant to mathematics (Sfard & Prusak, 2005). Sagor and Cox (2004) have identified five essential feelings they believe are crucial to a young person's well-being and success at school: 'the need to feel competent, the need to feel they belong, the need to feel useful, the need to feel potent, and the need to feel optimistic' (p. 4). They explain why working only on the behaviours and attitudes of discouraged learners is insufficient, and suggest including an additional dimension, the need to feel they have a valued role to play in the social context.

Consider and discuss your teaching

ROLE OF AFFECT IN LEARNING

STOP! Before you read the sentences below, cover them with a piece of paper. Slowly slide the paper down, read one sentence at a time, and then record your emotional reaction at the end of each sentence.

The baby kicked the ball.

The football player kicked the ball.

The golfer kicked the ball.

1 What did you notice?
2 What are the implications of this for teaching and learning mathematics?

Chances are that after the first sentence you experienced a warm feeling: 'Oh, how cute.' After the second sentence, you may have had a more neutral reaction, depending on your love or otherwise of football. But if you know anything at all about the rules of golf, you undoubtedly would have reacted with some indignation at the thought of a golfer kicking the ball.

Handy hint

Use rich mathematical tasks and investigations in mixed-ability groups to establish a classroom culture that ensures all students have an opportunity to participate and share their strengths with their peers.

Think + link: The issue of streaming will be considered in Chapter 8. Chapter 29 explores what is involved in becoming a reflective practitioner and what is involved in the broader work of a teacher of mathematics.

Teaching does not take place in isolation. Mandated curriculum, national testing, parent expectations and public perceptions about the capacities of school leavers all exert pressure on education systems, schools and teachers. This can lead to poor practices such as 'teaching to the test', over-scaffolding learning, and ability-based organisational structures (otherwise known as streaming or tracking), so teachers need to have the knowledge and confidence to withstand such pressures while meeting reasonable institutional requirements. Collaborative planning and a commitment to ongoing professional learning are key to maintaining quality in mathematics teaching and learning.

Conclusion

A deep understanding of mathematics is necessary for responsible citizenship in an increasingly globalised world. What we know and believe about the nature and purpose of school mathematics determines what we value and what we assess. It also has important implications for how we teach and what we expect of students as they engage with mathematics. The citizen of tomorrow will need to be able to mathematise, that is, to make sense of the world using mathematics through modelling and problem solving, rather than regurgitate a set of arid facts and procedures.

School mathematics is changing in response to research about the nature of children's learning of mathematics and studies of effective classroom practice, but it is also changing with advances in technology. If you were born before 1990, your school might have had one computer per classroom, but that computer would have had significantly less memory and processing capacity than a modern mobile phone, and it was far less user-friendly than an MP3 player. Digital technologies are changing how we learn and how we think about teaching. We now recognise that students need to have a deep understanding of how numbers work; access to a wide range of strategies that they can use confidently and flexibly to reason mathematically and solve problems; and the capacity to examine quantitative and spatial information critically.

Our collective knowledge of how mathematics is learnt has increased over the last 40 to 50 years. Before the 1950s, mathematics education was not recognised as a field of academic inquiry in English-speaking countries. The study of learning was very much the province of psychology, itself an emerging field, governed strictly by the rules of scientific experimentation, where rats in mazes and pigeons pecking levers were used as a basis for making claims about human learning. We now know that learning mathematics is a much more complex affair. While there is a place for perceptual learning—for instance, we need to recognise the number names and symbols—beyond this, most learning in mathematics needs to involve conceptual understanding; that is, it needs to build on meaningful ideas and multiple representations and be supported by collaborative discussion, rich and challenging tasks, and personal success.

As a consequence, what it means to be a teacher of mathematics has changed and is changing. We can no longer justify the 'sage on the stage' approach, where the teacher is seen as the fount of all wisdom, where power differentials are routinely exercised and one voice, the teacher's voice, dominates the discourse. Teachers are now expected to be orchestrators of engaging, purposeful learning environments who have the knowledge and confidence to make connections between important mathematical ideas and strategies and between individual learners and mathematics in-the-moment. It is critically important for teachers to understand the students they teach and the social context in which teaching occurs.

Review questions

1 Many people believe mathematics is an exact science where an answer is either right or wrong. Do you agree? Why? Can you find an instance where this is not the case?
2 What do you believe to be the principal goal of school mathematics? Consider this same question from the perspective of students, parents, employers and politicians. What differences, if any, might you expect?
3 Who is likely to be advantaged and disadvantaged in learning school mathematics? Why?
4 Write a maths autobiography about your experience as a student of school mathematics. Which teacher stands out? Why? Share your observations and make a list of the attributes of effective teachers of mathematics. Describe the sort of mathematics teacher you aspire to be.
5 How students feel about mathematics affects the way they participate and learn. Find out about mathematics anxiety. What are the likely causes, and how might it be avoided?
6 At the outset, we asked you to consider when you last 'truly, madly, deeply' enjoyed doing some mathematics. Describe your experience to a friend and consider the implications of this for teaching mathematics.

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Websites

http://timss.bc.edu/timss2011/

TIMSS reports every four years on the achievement of fourth and eighth grade students in more than 40 countries.

www.acara.edu.au/default.asp

This is where the latest version of the *Australian Curriculum: Mathematics* can be found together with examples of what is expected at different year levels.

www.aamt.edu.au

The Australian Association of Mathematics Teachers site provides information about latest trends, quality resources and professional learning opportunities in mathematics education.