OXFORD IB DIPLOMA PROGRAMME

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

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OXFORD

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Digital contents

Digital content overview

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Syllabus coverage

This book covers all the content of the Mathematics: analysis and approaches HL course. Click on this icon here for a document showing you the syllabus statements covered in each chapter.

Practice exam papers

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Worked solutions

Click on this icon here for worked solutions for all the questions in the book

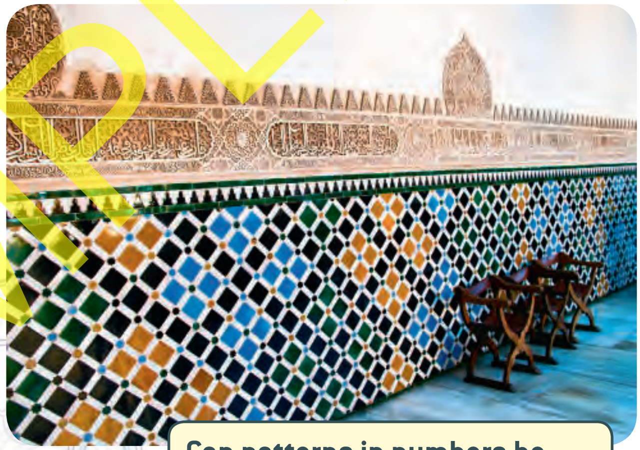
1 From patterns to generalizations: sequences, series and proof

Concepts
- Patterns
- Generalization

Microconcepts
- Arithmetic and geometric sequences and series
- Introduction to limits
- Sum of series
- Permutations and combinations
- Proof
- Binomial theorem

You do not have to look far and wide to find visual patterns—they are everywhere!

Can these patterns be explained mathematically?

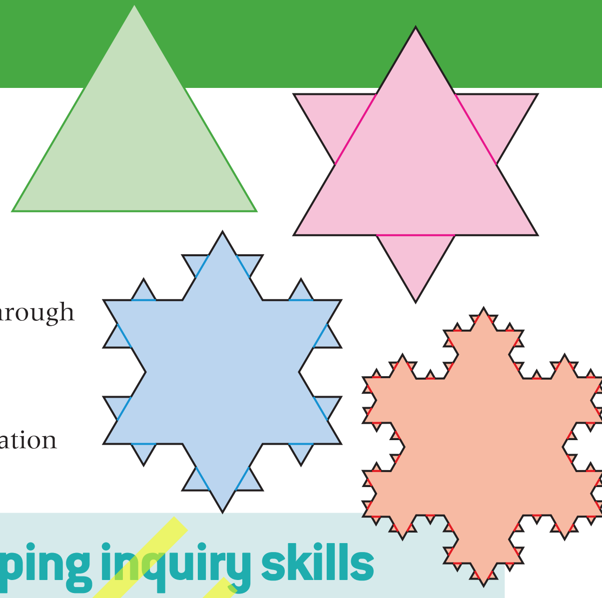


Can patterns in numbers be useful in real-life situations?

What information would you require to choose the best loan offer? What other scenarios could this be applied to?

If you take out a loan to buy a car, how can you determine the total amount it will cost?

The diagrams shown here are the first four iterations of a fractal called the Koch snowflake.



What do you notice about:

- How each pattern is created from the previous one?
- the perimeter as you move from the first iteration through the fourth iteration? How is it changing?

What changes would you expect in the fifth iteration?

How would you measure the perimeter at the fifth iteration if the original triangle had sides of 1m in length?

What happens if you start with a square instead of an equilateral triangle?

If this process continues forever, how can an infinite perimeter enclose a finite area?

Developing inquiry skills

Does mathematics always reflect reality? Are fractals such as the Koch snowflake invented or discovered?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

1 Solve linear algebraic equations.

eg $x - 3(x+5) = 20 - 3x$

$\Rightarrow x - 3x - 15 = 20 - 3x$

$\Rightarrow -2x - 15 = 20 - 3x$

$\Rightarrow x = 35$

2 Simplify surds.

eg simplify $\dfrac{\sqrt{2}}{1-\sqrt{2}}$

$$\frac{\sqrt{2}}{1-\sqrt{2}} = \frac{\sqrt{2}\left(1+\sqrt{2}\right)}{\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)} = \frac{\sqrt{2}+2}{1-2} = -2-\sqrt{2}$$

3 Manipulate algebraic fractions.

eg simplify $\dfrac{x+3}{x} = \dfrac{2}{x+1} - \dfrac{3x}{x-1}$

$$= \frac{(x+3)(x+1)(x-1)+2x(x-1)-3x^2(x+1)}{x(x+1)(x-1)}$$

$$= \frac{(x+3)(x^2-1)+2x^2-2x-3x^3-3x^2}{x(x^2-1)}$$

$$= \frac{x^3-x+3x^2-3+2x^2-2x-3x^3-3x^2}{x(x^2-1)}$$

$$= \frac{-2x^3+2x^2-3x-3}{x(x^2-1)}$$

Skills check

Click here for help with this skills check

1 Solve the following equations:

a $3x + 5(x-4) = 20x + 4$

b $\dfrac{x+1}{2x-1} = \dfrac{x-3}{2x+1}$

2 Simplify the following:

a $\dfrac{1+\sqrt{2}}{1-\sqrt{2}}$

b $\dfrac{2\sqrt{2}}{1-\sqrt{3}}$

3 Simplify:

$$\frac{x}{x+1} - \frac{1}{2x-1} + \frac{2}{x-1}$$

1.1 Sequences, series and sigma notation

Opening investigations

You are going to start this chapter by doing some simple arithmetic with the aim of recognizing patterns. The challenge is for you to understand and explain the patterns that emerge. In Investigation 2, you will be asked to propose a conjecture, which is a rule generalizing findings based on observed patterns.

Investigation 1

Work out the following products:

$1 \times 1 \qquad 11 \times 11 \qquad 111 \times 111 \qquad 1111 \times 1111$

1 What pattern do you see emerging?

2 Does this continue as you make the string of 1's longer?

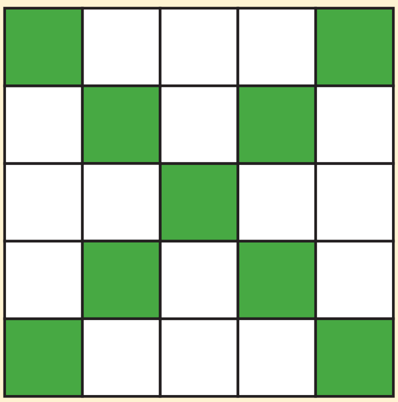
3 Can you predict when this pattern stops and explain why this happens?

International-mindedness

Where did numbers come from?

Investigation 2

This diagram represents the floor of a room covered with square tiles. It has a total of nine tiles along the main diagonals (shaded), and five tiles on each side. 25 tiles are used to cover the floor completely.



Another room has a total of 13 square tiles along the diagonals.

1 How many square tiles are there on each side in this other room?

2 How many tiles are needed to completely cover the floor?

3 What if the total number of tiles along the diagonals is 15?

4 What if there is a total of 135 tiles along the diagonals?

5 What if the total number of squares along the diagonals is an even number?

6 Continue to generate data to help you form a conjecture. Can you explain why this rule holds true?

7 How can you write the generalization concisely?

8 Why is an algebraic expression more useful than generating numerical values?

A **sequence** is a list of numbers that is written in a defined order, ascending or descending, following a specific rule. Each of the numbers making up a sequence is called a **term** of that sequence. Sometimes a sequence is also referred to as a **progression**.

Look at the following sequences of numbers and identify the rule which would help you obtain the next term.

i 7, 5, 3, 1, …

ii 2, 4, 8, 16, …

iii 1, 3, 9, 27, …

> Sequences may be **finite** or **infinite**.

The sequence 7, 5, 3, 1, −1, −3 is a finite sequence with six terms, whereas the sequence 7, 5, 3, 1, −1, −3, … is an infinite sequence with an infinite number of terms. The distinction is indicated by the ellipsis (…) at the end of the sequence.

A sequence is sometimes written in terms of the general term as $\{u_r\}$, where r can take values 1, 2, 3, …
If the sequence is finite then r will terminate at some point.

The sequence $\{u_r\} = \{3r - 1\}$, where $r \in \mathbb{Z}^+$ represents the infinite sequence 2, 5, 8, 11, …, whereas the sequence $\{u_r\} = \left\{\frac{1}{r^2}\right\}$, where $r \in \mathbb{Z}^+$, $r \leq 5$, represents the finite sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}$.

HINT

$\{u_r\}$ represents the sequence whereas u_r represents the rth term.

All the terms in a sequence added together are called a **series**. Like sequences, series can be finite or infinite.

The series obtained by adding the six terms of the sequence 7, 5, 3, 1, −1, −3 is $7 + 5 + 3 + 1 - 1 - 3 = 12$. This is a finite series. The sum $1 + 3 + 9 + 27 + 81 + \ldots$ continues indefinitely and is an infinite series.

The set of positive integers $\mathbb{Z}^+$ can be written as $\{1, 2, 3, 4, 5, \ldots, r, \ldots\}$ where the letter r is used to represent the general term. If the positive integers which are multiples of 5 are considered, then the set $\{5, 10, 15, 20, \ldots 5r, \ldots\}$ is obtained. In this case the general term is $5r$ where r is any positive integer. The **harmonic series** is the infinite sum of the reciprocals of positive integers, ie $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r} + \ldots$

TOK

Do the names that we give things impact how we understand them?

Series can be represented in compact form using sigma (Σ) notation. This makes use of the general term written in terms of r, which often represents a positive integer.

The sum of the first 10 positive integers can be written as follows using sigma notation:

$$\sum_{r=1}^{10} r$$

The largest value that r can take

The smallest value that r can take

Read this as "The sum of r, from $r = 1$ to $r = 10$."

If you want to write the sum of the positive multiples of 5 less than 100, then you first need to think of the general term, which is $5r$, and then establish the range of values that r can take. The smallest positive multiple of 5 is 5 in which case $r = 1$, and since you want the largest multiple of 5 to be 100, the largest value that r can take is 20 because $100 = 5 \times 20$.

$$5+10+15+\ldots+100=\sum_{r=1}^{20} 5r$$

Sometimes you will also have to interpret a sum given in sigma notation and expand it into individual terms. For example:

$$\sum_{r=0}^{4}(2r+1)=(2\times0+1)+(2\times1+1)+(2\times2+1)+(2\times3+1)+(2\times4+1)=1+3+5+7+9$$

HINT

In this case the series starts $r=0$.

In Example 1 you will learn how to look for a pattern and write the general term.

Example 1

For each of the following sequences, write the next three terms and find the general term:

a 2, 7, 12, 17, … **b** 2, 6, 12, 20, … **c** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ **d** 5, 10, 20, 40, …

a The next three terms of this sequence are 22, 27, 32. The sequence can be written as: $2, 2+5, 2+10, 2+15$ $= 2, 2+(1\times5), 2+(2\times5), 2+(3\times5), \ldots, 2+(r-1)\times5$ The general term is $2+(r-1)\times5 = 5r-3$, where r can take the values 1, 2, 3, …	Note that at each step you add 5 to get the next term. Write the sequence using the pattern noticed.
b The next three terms are 30, 42, 56. The sequence can be written as $1\times2, 2\times3, 3\times4, 4\times5, \ldots, r\times(r+1), \ldots$ The general term is $r\times(r+1)$, where r can take the values 1, 2, 3, …	Note that the given terms can be written as: $1\times2, 2\times3, 3\times4, 4\times5, \ldots$
c The next three terms are $\frac{5}{6}, \frac{6}{7}, \frac{7}{8}$. The sequence can be written as: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{r}{r+1}, \ldots$ The general term is $\frac{r}{r+1}$, where r can take the values 1,2,3, …	The pattern here is easy to follow.
d The next three terms are 80, 160, 320. The general term is $5\times2^{r-1}$, where r can take the values 1, 2, 3, …	Each term is obtained by multiplying the previous term by 2.

HINT

You can check the answers by putting $r = 5,6,7$ in the general term obtained in each case.

Example 2 shows how to find the terms of a sequence represented by its general term.

Example 2

Write down the first three terms of each of the following sequences:

a $\{u_r\} = \{5r - 2\},\ r \in \mathbb{Z}^+$ **b** $\{u_r\} = \left\{\frac{(-1)^r}{r^2}\right\},\ r \in \mathbb{Z}^+$

a $u_1 = 5 \times 1 - 2 = 3$ $u_2 = 5 \times 2 - 2 = 8$ $u_3 = 5 \times 3 - 2 = 13$ 3, 8, 13	Substitute values 1, 2 and 3 for *r*.
b $u_1 = \frac{(-1)^1}{1^2} = -1$ $u_2 = \frac{(-1)^2}{2^2} = \frac{1}{4}$ $u_3 = \frac{(-1)^3}{3^2} = -\frac{1}{9}$ $-1,\ \frac{1}{4},\ -\frac{1}{9}$	Substitute values 1, 2 and 3 for *r*.

Example 3 shows how to represent a given sequence by its general term after recognizing a pattern.

Example 3

Write each of the following sequences using the general term:

a 3, 6, 9, 12, … **b** 2, −10, 50, −250 **c** $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots$

a 3, 6, 9, 12, … $\{u_r\} = \{3r\},\ r \in \mathbb{Z}^+$	This is an infinite sequence of the positive multiples of 3.
b 2, −10, 50, −250 $\{u_r\} = \{2(-5)^{r-1}\},\ r \in \mathbb{Z}^+,\ r \le 4$	This finite sequence can be written as: $2,\ 2\times(-5),\ 2\times 25,\ 2\times(-125)$ which can be rewritten in terms of powers of −5: $= 2\times(-5)^0,\ 2\times(-5)^1,\ 2\times(-5)^2,\ 2\times(-5)^3$
c $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots$ $\{u_r\} = \left\{\frac{r}{2r+1}\right\},\ r \in \mathbb{Z}^+$	In this infinite sequence, the numerators are the positive integers and the denominators are successive odd integers greater than 1.

Example 4 shows how to expand a series written in sigma notation.

Example 4

For each of the following series written in sigma notation, write the first five terms:

a $\sum_{r=1}^{10} r(r-1)$ **b** $\sum_{r=1}^{\infty} (-1)^r r^2$ **c** $\sum_{r=1}^{\infty} \frac{r+1}{2r-1}$

a $\sum_{r=1}^{10} r(r-1) = 1\times 0 + 2\times 1 + 3\times 2 + 4\times 3 + 5\times 4 + \dots$

$= 0 + 2 + 6 + 12 + 20 + \dots$

Substitute $r = 1$ to 5 for the first through to the fifth term.

Simplify.

b $\sum_{r=1}^{\infty} (-1)^r r^2$

$= (-1)^1 \times 1^2 + (-1)^2 \times 2^2 + (-1)^3 \times 3^2$
$+ (-1)^4 \times 4^2 + (-1)^5 \times 5^2 + \dots$

$= -1 + 4 - 9 + 16 - 25 + \dots$

c $\sum_{r=1}^{\infty} \frac{r+1}{2r-1} = \frac{1+1}{2-1} + \frac{2+1}{4-1} + \frac{3+1}{6-1} + \frac{4+1}{8-1} + \frac{5+1}{10-1}$

$= 2 + 1 + \frac{4}{5} + \frac{5}{7} + \frac{6}{9} + \dots$

In Example 5 you will see how a given series can be written in sigma notation.

TOK

Is mathematics a language?

Example 5

Write each of the following series in sigma notation:

a $3 + 11 + 19 + 27 + 35$ **b** $1 - 1 + 1 - 1 + 1 - 1 + \dots$ **c** $-6 + 12 - 24 + 48 - 96 + 192$

a $3 + 11 + 19 + 27 + 35$

$= \sum_{r=1}^{5} 8r - 5$

This is a finite series which can be written as:
$3 + (3 + 8) + (3 + 16) + (3 + 24) + (3 + 32)$
$= 3 + (3 + 1 \times 8) + (3 + 2 \times 8) + (3 + 3 \times 8)$
$+ (3 + 4 \times 8)$

The general term is $3 + (r - 1) \times 8 = 8r - 5$.

b $1 - 1 + 1 - 1 + 1 - 1 + \dots$

$= \sum_{r=1}^{\infty} (-1)^{r-1}$

This is an infinite series. Each term oscillates between -1 and $+1$ and the general term is $(-1)^{r-1}$.

c $-6 + 12 - 24 + 48 - 96 + 192$

$\sum_{r=1}^{6} (-1)^r 6r$

This is a finite series with oscillating signs and each term is the next multiple of 6.

Exercise 1A

1 For each of the following sequences, write the next three terms and find the general term:

a $3,\ 4.5,\ 6,\ 7.5,\ \ldots$

b $17,\ 14,\ 11,\ 8,\ \ldots$

c $3,\ 9,\ 27,\ 81,\ \ldots$

d $\frac{1}{4},\ \frac{4}{7},\ \frac{7}{10},\ \frac{10}{13},\ \ldots$

e $\frac{1}{2},\ \frac{1}{12},\ \frac{1}{30},\ \frac{1}{56},\ \ldots$

2 Write down the first five terms of each of the following sequences:

a $u_r = 3 - 2r$

b $u_r = \frac{r}{2r+1}$

c $u_r = 2r + (-1)^r r$

d $u_r = (-1)^r \times 2$

e $u_r = \frac{3}{2^{r-1}}$

3 Write each of the following sequences using the general term:

a $5,\ 10,\ 15,\ 20,\ \ldots$

b $6,\ 14,\ 22,\ 30,\ \ldots$

c $\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots$

d $1,\ -\frac{1}{3},\ \frac{1}{9},\ -\frac{1}{27},\ \ldots$

e $0,\ 3,\ 8,\ 15,\ \ldots$

4 Write each of the following series in full:

a $\sum_{r=1}^{4} 2r(1-r)$

b $\sum_{r=0}^{5} (-1)^r r^2$

c $\sum_{r=1}^{5} \frac{r}{3r-1}$

d $\sum_{r=1}^{4} 5$

e $\sum_{r=0}^{3} (r^2 - 3)$

5 For each of the following series written in sigma notation, write the first five terms:

a $\sum_{r=1}^{\infty} \frac{r+1}{r^2}$

b $\sum_{r=1}^{\infty} \frac{(-1)^r}{2r^2-1}$

c $\sum_{r=1}^{20} r(5r-1)$

d $\sum_{r=0}^{5} (2^r - 3)$

e $\sum_{r=1}^{\infty} r^r$

6 Write each of the following series in sigma notation:

a $8 + 5 + 2 - 1 - 4$

b $3 + 10 + 21 + 36 + 55$

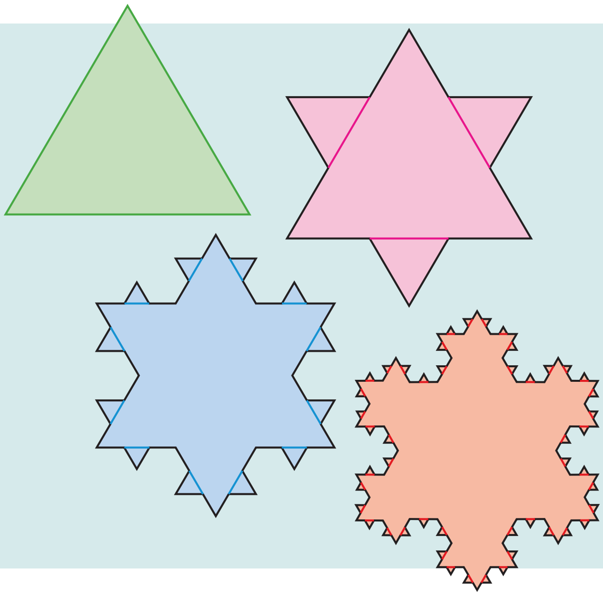
c $0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{2}{3} + \frac{5}{7} + \ldots$

d $1 + 9 + 25 + 49 + 81$

e $3k + 6k + 9k + 12k + 15k$

Developing inquiry skills

Now go back to the opening question. Suppose the length of each side of the first triangle is 81 cm. Can you work out the length of each side of the figure in each iteration? Tabulate your results and try to find a pattern and then make a conjecture.



1.2 Arithmetic and geometric sequences and series

Investigation 3

Whenever you go through airport security you have to place your hand luggage, coat, phone, etc into a tray that goes on a conveyer belt which then takes it through an x-ray scanner.

When answering the following questions, you can assume the following:

- Trays are placed on the conveyer belt with no gaps between them.
- The length of each tray is 60 cm.
- The conveyer belt is moving at 10 cm per second.
- Each person uses three trays.

1 Copy and complete the following table:

Number of people ahead of you	Distance of your first tray to machine, d (m)	Waiting time, T (s)
0	0	0
1	1.8	
2		36
⋮	⋮	⋮
n		

2 What patterns do you see emerging?

3 Now assume that there is a 30 cm gap separating trays belonging to different passengers. Construct and complete a table similar to the one above.

4 How have the patterns changed?

5 What happens if the distance between the trays of individual passengers changes to 50 cm? 60 cm? 80 cm?

6 How have the patterns changed?

7 **Factual** What do you notice about consecutive terms in the second and third columns?

8 **Factual** How would you generalize the relationship between the distance from the machine to your first tray and the number of people ahead of you?

9 **Factual** Write down the relationship between the waiting time and the number of people ahead of you.

10 **Conceptual** What common patterns generate the relationships developed in this investigation?

Arithmetic sequences and series

A growth pattern that is represented by a **linear relationship** is also known as an arithmetic sequence, which is defined as follows:

If the difference between two consecutive numbers in a sequence is constant then it is an **arithmetic sequence** or an **arithmetic progression**. The constant difference is called the **common difference** and is denoted by d.

Consider how an arithmetic sequence with first term u_1 and common difference d grows:

First term	u_1
Second term	$u_2 = u_1 + d$
Third term	$u_3 = u_2 + d = u_1 + 2d$
Fourth term	$u_4 = u_3 + d = u_1 + 3d$

This leads to the general term $u_n = u_1 + (n-1)d$.

HINT

A recursive equation is one in which the next term is defined as a function of earlier terms. In the case of an arithmetic sequence the recursive equation is $u_n = u_{(n-1)} + d$.

An arithmetic sequence with first term u_1 and common difference d has **general term** $u_n = u_1 + (n-1)d$.

The next four examples show you how to use the general term formula to answer different types of questions.

Example 6

The fourth term of an arithmetic sequence is 18 and the common difference is –5. Determine the first term and the nth term.

$u_4 = u_1 + 3 \times (-5) = 18$ $\Rightarrow u_1 = 18 + 15 = 33$ $u_n = 33 + (n-1) \times (-5)$ $\Rightarrow u_n = 38 - 5n$	Using $u_n = u_1 + (n-1)d$.

Example 7

Find the number of terms in the following arithmetic sequences:

a 20, 23, 26, …, 83 **b** 34, 30, 26, …, –30 **c** $6a, 4a, 2a, \ldots, -22a$

a $u_1 = 20,\ d = 3$ $u_n = 17 + 3n = 83$ $\Rightarrow n = 22$ **b** $u_1 = 34,\ d = -4$ $u_n = 38 - 4n = -30$ $\Rightarrow n = 17$ **c** $u_1 = 6a,\ d = -2a$ $u_n = 8a - 2an = -22a$ $\Rightarrow n = 15$	$d = 23 - 20 = 3$ Using $u_n = u_1 + (n-1)d$. Solve the linear equation to obtain n.

Example 8

Three numbers are consecutive terms of an arithmetic sequence. The sum of the three numbers is 45, and their product is 3240. Find the three numbers.

Let the three numbers be $u-d,\ u,\ u+d$	
$3u = 45$	Taking the sum of the numbers.
$\Rightarrow u = 15$	
$u(u^2 - d^2) = 3240$	Taking the product.
$\Rightarrow 15^2 - d^2 = \dfrac{3240}{15} = 216$	Substitute $u = 15$ and divide by 15.
$\Rightarrow d^2 = 225 - 216 = 9$	
$\Rightarrow d = \pm 3$	The two values of d produce two possible sequences:
	12, 15, 18 or
The three numbers are 12, 15 and 18.	18, 15, 12

Example 9

The second term of an arithmetic sequence is 20 and the seventh term is 55. Find the first term and the common difference of the sequence.

$u_2 = u_1 + d = 20$	$u_n = u_1 + (n-1)d$
$u_7 = u_1 + 6d = 55$	
$\Rightarrow 5d = 35 \Rightarrow d = 7$	
$u_1 = 20 - 7 = 13$	Solving simultaneously.
Or	
$u_7 = u_2 + 5d \Rightarrow 5d = 55 - 20$	Write u_7 in terms of u_2.
$\Rightarrow 5d = 35 \Rightarrow d = 7$	Solve for d.
$u_1 = 20 - 7 = 13$	

The sum of an arithmetic sequence

Investigation 4

Miss Sandra, the Grade 5 teacher, pairs up her students and gives each pair 55 cards numbered from 1 to 55. She tells the students that she wants them to use these cards to find the sum of the numbers $1 + 2 + 3 + \ldots + 55$.

Michela and Grisha start by laying out the cards in ascending order. Michela takes away the first card and the last card and notes that their sum is 56. Grisha then takes the first and last card from the cards that remain and notes that their sum is also 56. They continue to do this until just one card is left.

1 Which card will this be?
2 Using the information above, how would you determine the sum of the first 55 positive integers?
3 What if you wanted to find the sum of the first 1000 positive integers?
4 **Factual** Explain the importance of the actual number of terms added.
5 Repeat the process for finding the sum of:
   a the first 100 even numbers
   b the positive multiples of 3 less than 1000.
6 **Conceptual** How was Michela's and Grisha's method more efficient?

International-mindedness

Karl Friedrich Gauss (1777–1855) was a renowned German mathematician. It is said that when he was in primary school his teacher challenged him to find the sum of the numbers from 1 to 100. To the teacher's amazement, Gauss gave the correct answer almost immediately. He came to the answer by using the method used in investigation 4.

Reflect on Investigation 4 and explain how the method used is equivalent to the direct derivation for the sum of an arithmetic series containing n terms, with first term u_1 and common difference d as shown below.

$$\begin{array}{llllllllllll} S_n = u_1 & + & u_1+d & + & u_1+2d & + & \dots & + & u_1+(n-2)d & + & u_1+(n-1)d \\ S_n = u_1+(n-1)d & + & u_1+(n-2)d & + & u_1+(n-3)d & + & \dots & + & u_1+d & + & u_1 \\ \hline 2S_n = 2u_1+(n-1)d & + & 2u_1+(n-1)d & + & 2u_1+(n-1)d & + & \dots & + & 2u_1+(n-1)d & + & 2u_1+(n-1)d \end{array}$$

$$\Rightarrow 2S_n = n[2u_1+(n-1)d]$$

$$\Rightarrow S_n = \frac{n}{2}[2u_1+(n-1)d]$$

This can be rewritten as follows:

$$\begin{aligned} S_n &= \frac{n}{2}[2u_1+(n-1)d] \\ &= \frac{n}{2}[u_1+u_1+(n-1)d] \\ &= \frac{n}{2}[u_1+u_n] \end{aligned}$$

TOK

How is intuition used in mathematics?

The sum of a finite arithmetic series is given by $S_n = \frac{n}{2}[2u_1+(n-1)d] = \frac{n}{2}[u_1+u_n]$ where n is the number of terms in the series, u_1 is the first term, d is the common difference and u_n is the last term.

Number and algebra

Example 10

The first term of an arithmetic series is 5 and the last term is –51. The series has 15 terms. Find:

a the common difference

b the sum of the series.

a $-51 = 5 + 14d$ $d = \frac{-56}{14} = -4$	Using $u_n = u_1 + (n-1)d$.
b $S_{15} = \frac{15}{2}[5 + (-51)] = -345$	Using $S_n = \frac{n}{2}[u_1 + u_n]$.

Example 11

The first term of an arithmetic series is –7 and the fourth term is 23. The sum of the series is 689. Find the number of terms in the series.

$u_1 = -7$ $u_1 + 3d = 23 \Rightarrow d = \frac{23+7}{3} = 10$	Using $u_4 = u_1 + 3d$.
$S_n = 689 = \frac{n}{2}[-14 + (n-1)\times 10]$ $\Rightarrow 10n^2 - 24n - 1378 = 0$ $\Rightarrow 5n^2 - 12n - 689 = 0$ $\Rightarrow (5n+53)(n-13) = 0$ $\Rightarrow n = 13$, since $n \in \mathbb{Z}^+$	Rearrange and solve for n.

Reflect Why can n not be a rational or a negative number?

Example 12

Find the value of $\sum_{r=1}^{28} 5r - 4$.

$u_1 = 1$ $u_{28} = 140 - 4 = 136$	Substitute $r = 1$ and $r = 28$ to find the first and last terms.
$S_{28} = \frac{28}{2}(1 + 136) = 1918$	Using the formula $S_n = \frac{n}{2}[u_1 + u_n]$.

Example 13

The sum of an arithmetic series is given by $S_n = n(2n - 3)$. Find the common difference and the first three terms of the series.

$S_1 = u_1 = -1$	Using $S_n = n(2n - 3)$.
$S_2 = u_1 + (u_1 + d) \Rightarrow -2 + d = 2$	Using $S_2 = u_1 + u_2$.
$d = 4$	
$u_1 = -1$	
$u_2 = 3$	
$u_3 = 7$	

Exercise 1B

1 Find the nth term of each of these sequences:

a 3, 8, 13, 18, …

b 101, 97, 93, 89, …

c $a - 3, a + 1, a + 5, a + 9, \ldots$

d −20, −5, 10, 25, …

2 Find the terms indicated in each of these arithmetic sequences:

a 5, 11, 17, 23, … 15th term

b 10, 3, −4, −11, … 11th term

c $a, a + 2, a + 4, a + 6, \ldots$ 17th term

d 16, 12, 8, 4, … $(n + 1)$th term

3 Find the number of terms in each of these arithmetic sequences:

a 16, 11, 6, …, −64

b −108, −101, −94, …, 60

c −15, −19, −23, …, −95

d $2a + 5, 2a + 3, 2a + 1, \ldots, 2a - 23$

4 Determine the first term and the common difference of the arithmetic sequences that are generated by each of the following nth terms:

a $u_n = 5n - 7$

b $u_n = 3n + 11$

c $u_n = 6 - 11n$

d $u_n = 2a + 2n + 1$

5 The sixth term of an arithmetic sequence is 37 and the common difference is 7. Find the first term and the nth term.

6 The fifth term of an arithmetic sequence is 0 and the 15th term is 180. Find the common difference and the first term.

7 The sum of three consecutive terms of an arithmetic sequence is 24 and their product is −640. Find the three numbers.

8 Jung Ho earned €38 000 when he started his first job in the year 2000. He received a raise of €500 each consecutive year. Determine how much he earned in 2017? Evaluate in which year he would earn 50% more than his original salary for the first time.

9 Find the value of each of the following series:

a $3 - 3 - 9 - 15 - 21 - \ldots - 93$

b $31 + 40 + 49 + \ldots + 517$

c $(a - 1) + (a + 2) + (a + 5) + \ldots + (a + 146)$

10 Find the value of each of the following sums:

a $\sum_{r=1}^{50}(3r - 8)$

b $\sum_{r=1}^{100}(7 - 8r)$

c $\sum_{r=1}^{20}(2ar - 1)$, where a is a constant

11 Find the sums of the following sequences up to the term indicated:

a 4, −1, −6, … 15th term

b 3, 11, 19, … 10th term

c 1,−4, −9, … 20th term

12 Calculate the sum of an arithmetic series with 25 terms given that the fifth term is 19 and 10th term is 39.

13 The third term of an arithmetic sequence is −8, and the sum of the first 10 terms of the sequence is −230. Find:

a the first term of the sequence

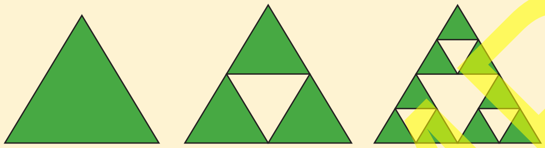
b the sum of the first 13 terms.

14 The sum of an arithmetic series is given by $S_n = 6n - 3n^2$. Find the common difference and the first four terms of the series.

15 Calculate the sum of all the odd numbers less than 300.

Investigation 5

The diagram below shows the first two iterations when constructing Sierpinski's triangle, named after the Polish mathematician Waclaw Sierpinski who first described it in 1915.



1 Construct the next iteration (Stage 3).

2 Copy and fill out the table below by following these instructions:

- Count the number of green triangles at each stage.
- If the sides of the triangle in stage 0 are each 1 unit long, what are the lengths of the sides of the green triangles at each of the following three stages? (Express your answers as rational numbers.)
- Now assume that the area of the triangle at Stage 0 is 1 unit². What is the area of each green triangle at each of the next three stages? (Leave answers in fractional form.)

Stage	0	1	2	3
Number of green triangles	1			
Length of one side of one green triangle	1			
Area of each green triangle	1			

3 **Factual** What patterns emerge from each of the three rows of the table?

4 **Factual** What do these three patterns have in common?

5 Based on your results, form a conjecture to obtain the numbers if you were to extend the table further to stages 4, 5, 6, etc.

6 **Conceptual** How would you compare the sets of numbers obtained?

TOK

Is all knowledge concerned with identification and use of patterns?

Geometric sequences and series

In Investigation 5 you should have noticed that when filling out the table you would need to multiply the numbers in each row by a particular constant to obtain the following column. In other words, the ratio of a particular term to the previous term is a constant. Such sequences are known as geometric sequences.

> If the ratio of two consecutive terms in a sequence is constant then it is a **geometric sequence** or a **geometric progression**. The constant ratio is called the **common ratio** and denoted by r.

HINT

The recursive equation for a geometric sequence is $u_n = u_{n-1} \times r$.

Consider how a geometric sequence with first term u_1 and common ratio r grows:

First term	u_1
Second term	$u_2 = u_1 r$
Third term	$u_3 = u_2 r = u_1 r^2$
Fourth term	$u_4 = ur = u_1 r^3$

This leads to the general term $u_n = u_1 r^{n-1}$.

> A geometric sequence with first term u_1 and common ratio r has **general term** $u_n = u_1 r^{n-1}, r \neq 1, 0, -1, u_1 \neq 0$.

Curiosities in geometric patterns

- What happens if you have a sequence with first term u_1 and common ratio 1?
- What if the common ratio is 0?
- And what happens if the common ratio is −1?

In the first case, the sequence is just made up of constant terms u_1. This is called a uniform sequence.

The next case is a sequence with first term u_1 and all the other terms are 0, which is a rather uninteresting sequence.

The third case leads to what is known as an oscillating sequence: $u_1, -u_1, u_1, -u_1, \ldots$

This oscillating sequence becomes particularly interesting if $u_1 = 1$, which then leads to the sequence 1, −1, 1, −1, 1, …

If you try to take the sum of this series you run into some curious and interesting results.

You want to look at the sum $S = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$

There are various ways of looking at this sum. Possibly the most intuitive way of finding this sum is by grouping the terms into pairs as follows:

$$S = (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0 + 0 + 0 + 0 + \ldots = 0$$

But what happens if you pair the terms starting from the second term instead of the first?

$$\begin{aligned} S &= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots \\ &= 1 + 0 + 0 + 0 + 0 \ldots = 1 \end{aligned}$$

Yet another result is obtained if you look at the series from a different perspective:

$$\begin{aligned} S &= 1 - (1 - 1 + 1 - 1 + 1 - 1 + \ldots) \\ &= 1 - S \\ \Rightarrow 2S &= 1 \\ \Rightarrow S &= \frac{1}{2} \end{aligned}$$

Why does this paradox arise and which is the correct answer? You have once more stumbled on the concept of infinity. If the number of terms were to be made finite, then the result would be 0 if there are an even number of terms, and 1 if the number of terms were odd, but an infinite sum never ends.

The next examples show how to use the general term formula for a geometric sequence to answer different types of questions.

International-mindedness

The series $S = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$ is known as Grandi's series, after the Italian mathematician Guido Grandi (1671–1742). You may want to look into the history and research on this sum by various mathematicians after its first appearance in Grandi's book published in 1703.

Example 14

Find the common ratio and write the next two terms of each sequence:

a 2.5, 5, 10, … **b** 9, 3, 1, … **c** $x, 2x^3, 4x^5, \ldots$

a $r = \frac{5}{2.5} = 2$ The next two terms are 20, 40.	Find r by calculating $\frac{u_2}{u_1}$. Use the recursive equation to find the next two terms.
b $r = \frac{3}{9} = \frac{1}{3}$ The next two terms are $\frac{1}{3}, \frac{1}{9}$.	
c $r = \frac{2x^3}{x} = 2x^2$ The next two terms are $8x^7, 16x^9$.	

Example 15

Find the number of terms in each of these geometric sequences:

a 0.15, 0.45, 1.35, …, 12.15 **b** 440, 110, 27.5, …, 0.4296875

a $u_1 = 0.15,\ r = \dfrac{0.45}{0.15} = 3$

Determine the value of r by computing $\dfrac{u_2}{u_1}$.

$u_n = 0.15 \times 3^{n-1} = 12.15$

Use $u_n = u_1 r^{n-1}$ to find n.

$\Rightarrow 3^{n-1} = 81 = 3^4$

$\Rightarrow n = 5$

This sequence has 5 terms.

b $u_1 = 440,\ r = \dfrac{110}{440} = 0.25$

$u_n = 440 \times 0.25^{n-1} = 0.4296875$

$\Rightarrow n - 1 = 5$

$\Rightarrow n = 6$

This sequence has 6 terms.

Use technology to find the value of n.

x	$440 \times (0.25)^x$
1	110
2	27.5
3	6.875
4	1.71875
5	0.4296875
6	0.1074219
7	0.0268555

Example 16

The first term of a geometric sequence is 4 and the common ratio is –2. Determine which term has the value of –2048?

$4 \times (-2)^{n-1} = -2048$

$\Rightarrow n = 10$

Use technology to find the value of n.

x	$4 \times (-2)^{x-1}$
1	4
2	–8
3	16
4	–32
5	64
6	–128
7	256
8	–512
9	1024
10	–2048

HINT

This time the formula uses $(x - 1)$ in the exponent so the answer is $n = 10$.

Example 17

The fourth term of a geometric sequence is 54 and the sixth term is 486. Determine the possible values of the common ratio.

$\left.\begin{array}{l} u_4 = u_1 \times r^3 = 54 \\ u_6 = u_1 \times r^5 = 486 \end{array}\right\} \Rightarrow r^2 = \frac{486}{54} = 9$ $r = \pm 3$ **Or** $u_6 = u_4 \times r^2$ $\Rightarrow r^2 = \frac{486}{54} = 9$ $\Rightarrow r = \pm 3$	Use $u_n = u_1 r^{n-1}$. Divide the two expressions to obtain r^2.

Example 18

The first term of a geometric sequence is 16 and the common ratio is $\frac{1}{2}$.

Find the biggest term that is smaller than $\frac{1}{1000}$.

$$16 \times \left(\frac{1}{2}\right)^{n-1} < 0.001$$

$$\Rightarrow n = 15$$

$$u_{15} = 16 \times \left(\frac{1}{2}\right)^{14}$$

Alternatively, you can use your GDC.

Use technology to find the value of n.

x	$16 \times (0.5)^{(x-1)}$
1	16
2	8
3	4
4	2
.	.
.	.
13	0.00390625
14	0.001953125
15	0.000976563
16	0.000488281

The sum of a geometric sequence

When trying to find the value of the series $S = 1 + 3 + 9 + 27 + 81 + 243$, Max notices that this is a geometric series with common ratio 3, and that if he were to multiply the series by 3, he could more easily calculate the sum as follows:

$$\left.\begin{array}{r} 3S = 3 + 9 + 27 + 81 + 243 + 729 \\ S = 1 + 3 + 9 + 27 + 81 + 243 \end{array}\right\} \Rightarrow 3S - S = 2S = 729 - 1 = 728$$

$$S = 364$$

Max then tried to generalize this result for a finite geometric series with common ratio r and having n terms as follows:

$$\left.\begin{aligned} S_n &= u_1 + u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^{n-1} \\ rS_n &= u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^{n-1} + u_1 r^n \end{aligned}\right\} \Rightarrow (1-r)S_n = u_1 - u_1 r^n$$

$$\Rightarrow S_n = \frac{u_1(1-r^n)}{1-r}, \quad r \neq 1$$

HINT

This formula can also be written as follows:

$$S_n = \frac{u_1(r^n - 1)}{r-1}, \quad r \neq 1.$$

This makes calculations easier when $r > 1$.

The sum of a finite geometric series is given by

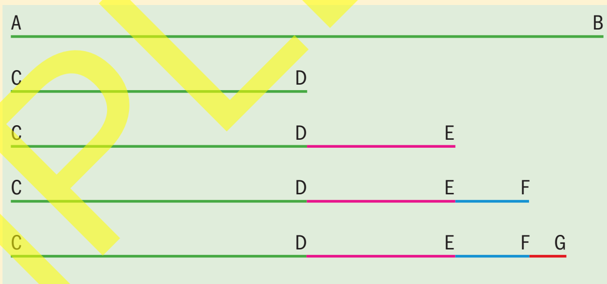
$$S_n = \frac{u_1(1-r^n)}{1-r}, \quad r \neq 1$$

where n is the number of terms, u_1 is the first term and r is the common ratio.

Investigation 6

In the diagram, AB represents a piece of string which is 100 cm long.

The string is cut in half and one of the halves, CD, is placed underneath. The remaining half is now cut in half and one of the halves, DE, is placed next to CD. The process is continued as shown in the diagram.



1 Copy and complete the table below.

Line segment	Length of string segment (cm)	Total length of segments (cm)
CD	50	50
DE	25	75
EF		
FG		

2 **Factual** As this process continues indefinitely, what do you notice about the length of the line segments? What about the total length of segments?

3 **Factual** What type of sequence is this?

Modelling this scenario mathematically:

$\mathrm{CD} = 50 \text{ cm}$

$$\mathrm{DE} = 50 \text{ cm} \times \frac{1}{2} = 25 \text{ cm}$$

$$\mathrm{EF} = \mathrm{DE} \times \frac{1}{2} = 50 \text{ cm} \times \left(\frac{1}{2}\right)^2 = 12.5 \text{ cm}$$

$$\mathrm{CD} + \mathrm{DE} + \mathrm{EF} + \mathrm{FG} = 50 + 50 \times \left(\frac{1}{2}\right) + 50 \times \left(\frac{1}{2}\right)^2 + 50 \times \left(\frac{1}{2}\right)^3$$

Continued on next page

After four cuts have been made the sum of the length of string segments placed next to each other is a geometric sequence with four terms. Show that if n cuts are made this sum becomes $\dfrac{50\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\left(\frac{1}{2}\right)}$.

Enter this into a table as shown below to see what happens as n gets bigger.

n	$\dfrac{50\times(1-(0.5)^n)}{(1-0.5)}$
1	50
2	75
3	87.5
4	
5	
6	
7	
8	

4 What would happen if you repeated this experiment, but this time you cut CD to be $\frac{2}{3}$ of AB and DE to be $\frac{2}{3}$ of the remaining piece of string?

5 Repeat the process using CD to be $\frac{3}{4}$ of AB and DE to be $\frac{3}{4}$ of the remaining piece of string. What if the fraction used was $\frac{4}{5}$?

6 Write a short reflection on your results which includes answers to the following questions:

- **Factual** Why were you asked to change the length of the string cut?
- **Conceptual** How has this process helped you analyse the situation?
- **Conceptual** How can the sum of an infinite series converge to a finite number?

Convergent and divergent series

An infinite geometric series is **convergent** when the sum tends to a finite value as the number of terms gets bigger. If a geometric series does not converge it is said to be **divergent**.

In Investigation 6, the series always converged to 100 cm, the length of the original piece of string.

Investigation 7

In Investigation 6, you would have noticed that you had a geometric series in each case. You will now investigate a general geometric series in order to understand which conditions will make the series converge.

For a geometric series, you know that $S_n = \dfrac{u_1(1-r^n)}{1-r},\ r \neq 1.$

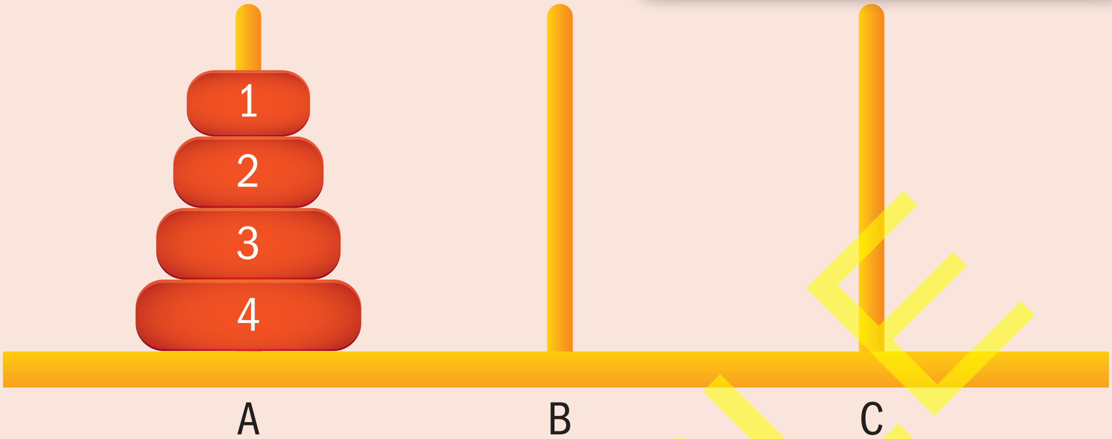
The Towers of Hanoi

Approaches to learning: Thinking skills, Communicating, Research

Exploration criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

IB topic: Sequences

The problem



The aim of the **Towers of Hanoi problem** is to move all the disks from peg A to peg C following these rules:

1 Move only one disk at a time.

2 A larger disk may not be placed on top of a smaller disk.

3 All disks, except the one being moved, must be on a peg.

For 64 disks, what is the **minimum** number of moves needed to complete the problem?

Explore the problem

Use an online simulation to explore the Towers of Hanoi problem for three and four disks.

What is the minimum number of moves needed in each case?

Solving the problem for 64 disks would be very time consuming, so you need to look for a rule for n disks that you can then apply to the problem with 64 disks.

Try and test a rule

Assume the minimum number of moves follows an arithmetic sequence.

Use the minimum number of moves for three and four disks to predict the minimum number of moves for five disks.

Check your prediction using the simulator.

Does the minimum number of moves follow an arithmetic sequence?

Find more results

Use the simulator to write down the number of moves when $n = 1$ and $n = 2$.

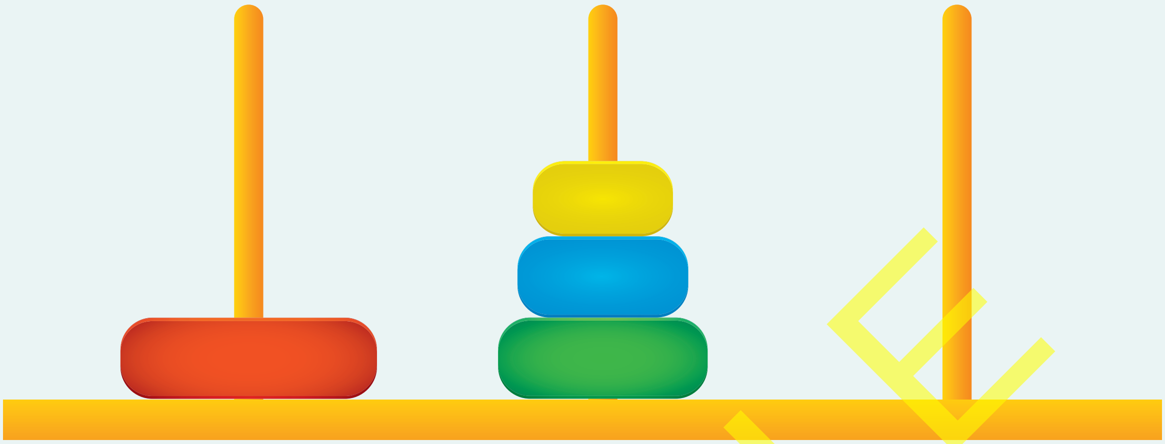
Organize your results in a table.

Look for a pattern. If necessary, extend your table to more values of n.

Try a formula

Return to the problem with four disks.

Consider this image of a partial solution to the problem. The large disk on peg A has not yet been moved.



Consider your previous answers.

What is the minimum possible number of moves made so far?

How many moves would it then take to move the largest disk from peg A to peg C?

When the large disk is on peg C, how many moves would it then take to move the three smaller disks from peg B to peg C?

How many total moves are therefore needed to complete this puzzle?

Use your answers to these questions to write a formula for the minimum number of moves needed to complete this puzzle with n disks.

This is an example of a **recursive formula**. What does that mean?

How can you check if your recursive formula works?

What is the problem with a recursive formula?

Try another formula

You can also try to solve the problem by finding an **explicit formula** that does not depend on you already knowing the previous minimum number.

You already know that the relationship is not arithmetic.

How can you tell that the relationship is not geometric?

Look for a pattern for the minimum number of moves in the table you constructed previously.

Hence write down a formula for the minimum number of moves in terms of n.

How does an explicit formula differ from a recursive formula?

Use your explicit formula to solve the problem with 64 disks.

Extension

- What would a solution look like for four pegs? Does the problem become harder or easier?
- Research the "Bicolor" and "Magnetic" versions of the Towers of Hanoi puzzle.
- Can you find an explicit formula for other recursive formulae? (eg Fibonacci)