15

Polynomials

This chapter deals with the algebra of polynomials and sketching polynomials.

After completing this chapter you should be able to:

- recognise polynomial expressions and perform algebraic operations with polynomials
- apply the remainder and factor theorems to polynomial functions
- sketch linear, quadratic and cubic polynomial functions
- determine the effect of single, double and triple roots of a polynomial equation on the shape of a curve.

NSW Syllabus references: 5.3 N&A Polynomials

Outcomes: MA5.3-1WM, MA5.3-2WM, MA5.3-3WM, MA5.3-10NA

Number & algebra – ACMNA266, ACMNA268

Diagnostic test

1 The expansion of $(x - 5)^2$ is:

A $x^2 - 25$ **B** $x^2 + 25$
C $x^2 - 5x + 25$ **D** $x^2 - 10x + 25$

2 The expansion of $(3x + 2)^2$ is:

A $3x^2 + 4$ **B** $9x^2 + 6x + 4$
C $9x^2 + 12x + 4$ **D** $9x^2 + 4$

3 The expansion of $(x - 2)(x + 2)$ is:

A $x^2 - 4$ **B** $x^2 + 4$
C $x^2 + 2x + 4$ **D** $x^2 - 4x + 4$

4 The expansion of $(5 - 2x)(5 + 2x)$ is:

A $4x^2 - 25$ **B** $25 - 4x^2$
C $25 - 10x + 4x^2$ **D** $25 - 20x + 4x^2$

5 The number needed to complete the square of $x^2 - 9x$ is:

A 9 **B** $\frac{9}{2}$
C $\frac{81}{4}$ **D** any number

6 When factorised $c^2 - a^2$ is:

A $(c - a)(c - a)$ **B** $(c - a)(c + a)$
C $(c - a)^2$ **D** $c^2 - 2ac + c^2$

7 When factorised $x^2 - x - 12$ is:

A $(x - 4)(x + 3)$ **B** $(x + 4)(x - 3)$
C $(x - 6)(x + 2)$ **D** $(x + 6)(x - 2)$

8 When factorised $10x^2 + 17x + 3$ is:

A $(5x + 3)(2x + 1)$ **B** $(5x - 1)(2x + 3)$
C $(5x + 1)(2x + 3)$ **D** $(5x - 1)(2x - 3)$

9 The solutions of $5x^2 = 3$ are:

A $x = \pm\frac{3}{5}$ **B** $x = \pm\sqrt{\frac{3}{5}}$
C $x = \pm\sqrt{15}$ **D** $x = \pm\sqrt{3}$

10 The solutions of $3x^2 + 9x = 0$ are:

A 0 and 3 **B** 0 and -3
C 0 and 9 **D** -3 and 9

11 The solutions of $x^2 + 7x - 18 = 0$ are:

A -9 and 2 **B** 9 and -2
C 6 and -3 **D** -6 and 3

12 The solutions of $3x^2 + 7x + 4 = 0$ are:

A 1 and 4 **B** -1 and -4
C -1 and $-\frac{4}{3}$ **D** 1 and $\frac{3}{4}$

13 The solutions of $\frac{x^2 - 12}{x} = 4$ are:

A -6 or 2 **B** 6 or -2
C 0 or $\sqrt{12}$ **D** -4 or -3

14 The constant term that needs to be added to solve $x^2 - 10x - 2 = 0$ by completing the square is:

A -5 **B** 5 **C** 25 **D** -25

15 The solutions to $x^2 - 5x + 3 = 0$ are:

A $\frac{-5 \pm \sqrt{13}}{2}$ **B** $\frac{5 \pm \sqrt{13}}{2}$
C $\frac{-5 + \sqrt{37}}{2}$ **D** $\frac{5 \pm \sqrt{37}}{2}$

16 The solutions to $4x^2 - 3x - 8 = 0$ are:

A $\frac{3 \pm \sqrt{137}}{8}$ **B** $\frac{-3 \pm \sqrt{94}}{8}$
C $\frac{-3 \pm \sqrt{137}}{4}$ **D** $\frac{3 \pm \sqrt{94}}{4}$

The diagnostic test questions refer to outcomes ACMMG233, ACMMG241 and ACMMG269.

A Curve sketching

For the following investigations students will require the use of a graphics calculator and/or graphics software. If these are not available, the use of a suitable table of values or a spreadsheet is recommended.

Investigation 1 The graph of $y = x^n$

1 a Sketch the following graphs by completing the table below and choosing a suitable scale.

i $y = x^2$ ii $y = x^3$ iii $y = x^4$ iv $y = x^6$

x	-2.5	-2.0	-1.5	-1.0	-0.5	0	0.5	1.0	1.5	2.0	2.5
y											

b Describe the difference between the shapes of graphs with an even index and those with an odd index.

c What are the x- and y-intercepts of each of the curves?

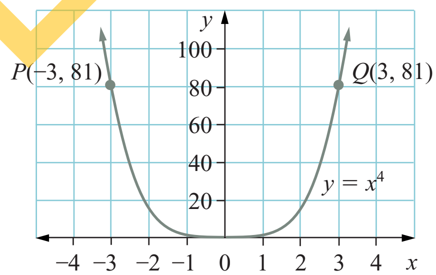
d What happens to the value of y as:

i $x \to +\infty$ (x gets very large and positive)? ii $x \to -\infty$ (x get very large and negative)?

e Is there a value of:

i y for every value of x? ii x for every value of y?

2 Consider the points $P(-3, 81)$ and $Q(3, 81)$ on the curve $y = x^4$. Join PQ and let N be the point where it cuts the y-axis.



a i Is PQ perpendicular to the y-axis?

ii Is $PN = NQ$? Give reasons.

b What transformation will map P onto Q?

c Can we always find pairs of points on this curve for which part **b** will be true? Give two examples.

d Hence, what kind of symmetry does this curve have?

3 Consider the points $P(-2, -8)$ and $Q(2, 8)$ on the curve $y = x^3$.

a Find the coordinates of the midpoint of the interval PQ.

b Draw a sketch showing the points P and Q and join POQ.

c What transformation will map P onto Q?

d Can we always find pairs of points on this curve for which part **c** will be true? Give two examples.

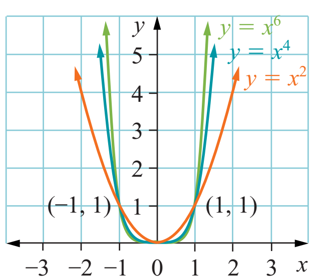
e Hence, what kind of symmetry does this curve have?

Summary

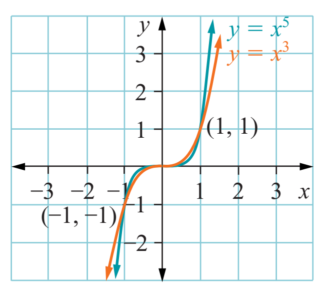
The graph of $y = x^n$ always passes through the origin. Graph A is **concave up** for all values of x and has line symmetry about the y-axis; that is, the y-axis is the axis of symmetry of the graph.

Graph B is **concave up** where $x > 0$ and **concave down** where $x < 0$. It has point symmetry (of order 2) about the origin; that is, any point on the graph when rotated through 180° about the origin will also lie on the graph.

Graph A: n is even.



Graph B: n is odd.



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Investigation 2 The graph of $y = ax^n$

1 On the same diagram, sketch graphs of:

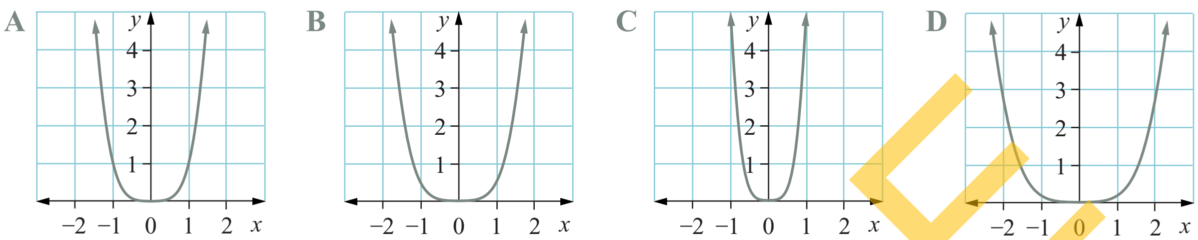
a i $y = x^2$ ii $y = 2x^2$ iii $y = 3x^2$ iv $y = \frac{1}{2}x^2$ v $y = \frac{1}{3}x^2$

b i $y = x^3$ ii $y = 2x^3$ iii $y = 3x^3$ iv $y = \frac{1}{2}x^3$ v $y = \frac{1}{3}x^3$

c Describe in words the effect of the constant a on the graph of $y = ax^n$, when compared with the graph of $y = x^n$.

2 Match each graph below with its correct equation.

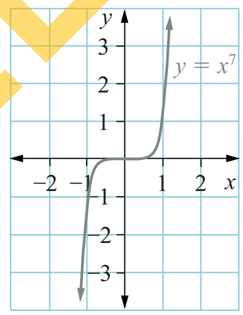
a $y = x^4$ **b** $y = 5x^4$ **c** $y = \frac{1}{2}x^4$ **d** $y = \frac{1}{6}x^4$



3 This is a sketch of $y = x^7$. Copy this graph and on the same axes draw the graphs of:

a $y = \frac{2}{3}x^7$ **b** $y = 3x^7$

c $y = \frac{5}{4}x^7$ **d** $y = 10x^7$



Summary

For the graph of $y = ax^n$, where n is a positive integer, the effect of the constant a is to:

- compress the curve $y = x^n$ horizontally and make it steeper when $a > 1$
- stretch the curve $y = x^n$ horizontally and make it less steep when $0 < a < 1$.

We will investigate negative values of a later in this chapter.

Investigation 3 The graph of $y = ax^n + k$

1 **a** On the same axes for $-2 \leqslant x \leqslant 2$, sketch graphs of:

i $y = 2x$ ii $y = 2x + 3$ iii $y = 2x - 3$

b Write the gradient and y-intercept of each line.

2 **a** On the same axes for $-2 \leqslant x \leqslant 2$, sketch graphs of:

i $y = 3x^2$ ii $y = 3x^2 + 2$ iii $y = 3x^2 - 1$

b For each curve:

i state the y-intercepts

ii state the equation of the axis of symmetry

iii find the coordinates of the vertex.

3 **a** On the same axes for $-2 \leqslant x \leqslant 2$, sketch graphs of:

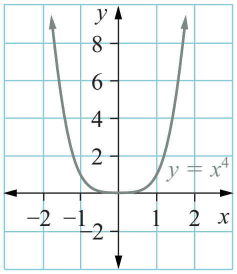
i $y = x^3$ ii $y = x^3 + 1$ iii $y = x^3 - 2$

b Write the y-intercept of each curve.

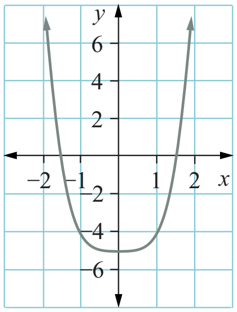
4 Given the graph of a function $y = ax^n$, describe in words how to sketch the graph of $y = ax^n + k$.

5 This is a sketch of the function $y = x^4$. Match each graph below with its correct equation.

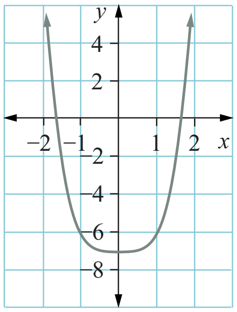
a $y = x^4 - 5$ **b** $y = x^4 + 1$
c $y = x^4 - 7$ **d** $y = x^4 + 5$



A



B



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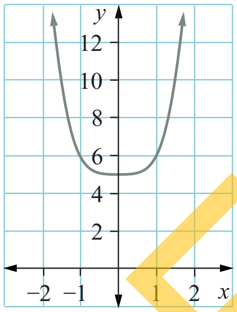
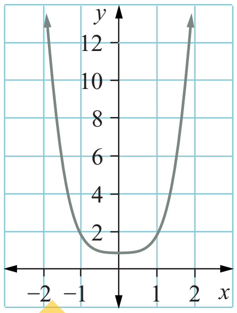
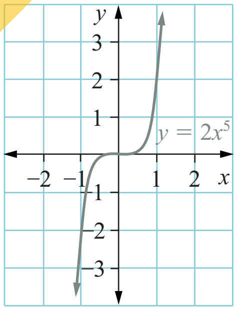


D



6 This is a sketch of $y = 2x^5$. Copy this graph and on the same axes draw the graphs of:

a $y = 2x^5 + 3$ **b** $y = 2x^5 - 1$



Summary

For $y = ax^n + k$, the effect of the constant k is to:

- translate the graph of $y = ax^n$ vertically k units up when $k > 0$
- translate the graph of $y = ax^n$ vertically k units down when $k < 0$.

Investigation 4 The graph of $y = a(x - b)^n$

1 a Sketch graphs of:
 i $y = 2x$ **ii** $y = 2(x - 1)$ **iii** $y = 2(x + 1)$

b Write the gradient and y-intercept of each line.

c Find the x-intercept of each line.

d What would be the gradient and x-intercept of $y = 2(x - 15)$?

2 a Sketch graphs of the following functions on the same axes.

i	$y = 3x^2$	for $-3 \leqslant x \leqslant 3$	**ii**	$y = 3(x - 1)^2$	for $-2 \leqslant x \leqslant 4$
iii	$y = 3(x - 2)^2$	for $1 \leqslant x \leqslant 5$	**iv**	$y = 3(x + 1)^2$	for $-4 \leqslant x \leqslant 2$
v	$y = 3(x + 2)^2$	for $-1 \leqslant x \leqslant 1$	**vi**	$y = 3(x - 5)^2$	for $2 \leqslant x \leqslant 8$

b For each curve find the:
 i x- and y-intercepts **ii** equation of the axis of symmetry
 iii coordinates of the vertex.

c What would be the coordinates of the vertex of each of these functions?
 i $y = 3(x - 14)^2$ **ii** $y = 3(x + 17)^2$

3 **a** Sketch graphs of the following functions on the same axes.

- **i** $y = 2x^3$ for $-2 \leqslant x \leqslant 2$
- **ii** $y = 2(x - 1)^3$ for $-1 \leqslant x \leqslant 3$
- **iii** $y = 2(x - 2)^3$ for $0 \leqslant x \leqslant 4$
- **iv** $y = 2(x + 1)^3$ for $-3 \leqslant x \leqslant 1$
- **v** $y = 2(x + 2)^3$ for $-4 \leqslant x \leqslant 0$

b What is the x-intercept for each of these curves?

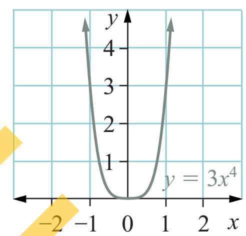
c Using only the results of parts **a** and **b**, can you sketch these graphs on your diagram?

- **i** $y = 2(x - 10)^3$
- **ii** $y = 2(x + 9)^3$

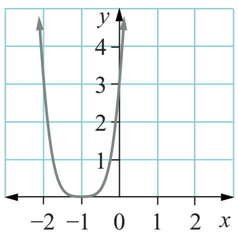
4 Given the graph of a function $y = ax^n$, describe how to sketch the graph of $y = a(x - b)^n$.

5 A sketch of the function $y = 3x^4$ is given. Match each graph below with its correct equation.

- **a** $y = 3(x - 1)^4$
- **b** $y = 3(x + 1)^4$
- **c** $y = 3(x + 3)^4$
- **d** $y = 3(x - 5)^4$



A



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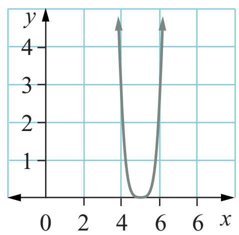

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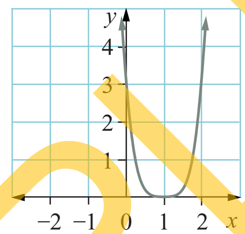
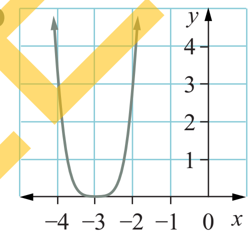


D



6 On the same axes draw sketches of:

- **a** $y = x^5$
- **b** $y = (x - 4)^5$
- **c** $y = (x + 3)^5$
- **d** $y = \left(x + \frac{3}{2}\right)^5$

Summary

For $y = a(x - b)^n$ the effect of the constant b on $y = ax^n$ is to:

- translate the graph of $y = f(x)$ horizontally b units to the right when $b > 0$
- translate the graph of $y = f(x)$ horizontally b units to the left when $b < 0$.

Investigation 5 The graph of $y = -f(x)$

1 On the same axes sketch the graphs of:

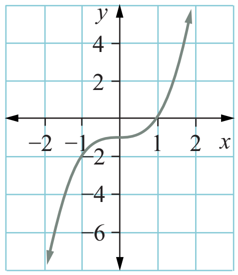
- **a** $y = 2x^3$ and $y = -2x^3$
- **b** $y = 3x^4$ and $y = -3x^4$
- **c** $y = x^3 + 1$ and $y = -(x^3 + 1)$
- **d** $y = 3(x - 1)^2$ and $y = -3(x - 1)^2$

2 Describe in words the relationship between the graphs of $y = f(x)$ and $y = -f(x)$.

3 Match each of the following graphs with its correct equation.

- **a** $y = 4x^2$
- **b** $y = -4x^2$
- **c** $y = x^3 - 1$
- **d** $y = -x^3 + 1$
- **e** $y = (x + 2)^6$
- **f** $y = -(x + 2)^6$

A



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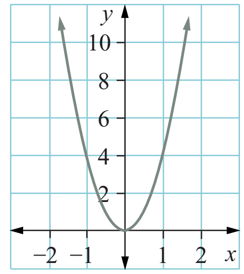

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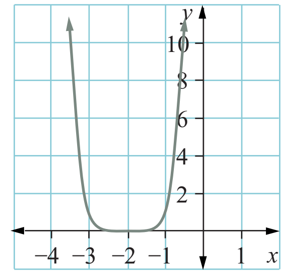

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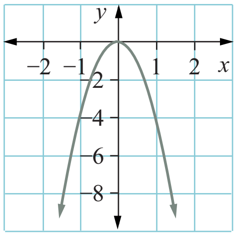
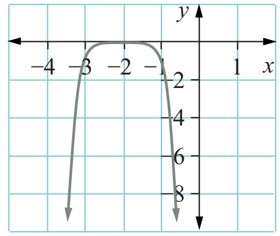
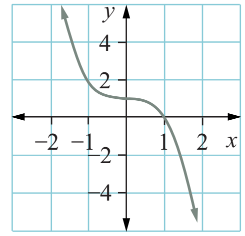
D



E



F



4 On the same diagram, draw freehand sketches of:

a $y = x^4$ and $y = -x^4$ **b** $y = 5x^2$ and $y = -5x^2 + 7$ **c** $y = (x - 2)^3$ and $y = -(x - 2)^3$

Summary

The graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ in the x-axis.

Exercise 15A

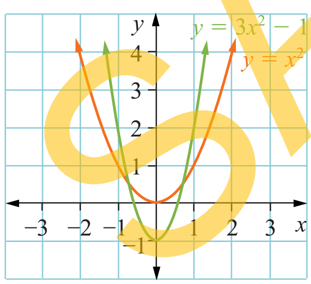
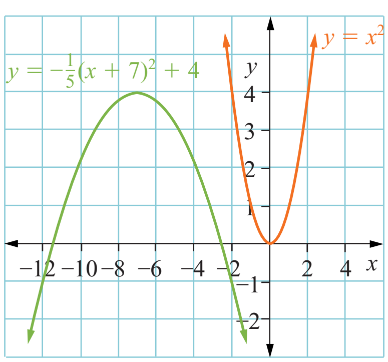
1 Sketch graphs of the following functions on the same axes.

a	**i** $y = x^2$	**ii** $y = 4x^2$	**iii** $y = x^2 + 4$	**iv** $y = 4x^2 - 1$	**v** $y = (x - 4)^2$
b	**i** $y = x^3$	**ii** $y = x^3 - 8$	**iii** $y = (x - 8)^3$	**iv** $y = 2(x - 8)^3$	**v** $y = (x - 8)^3 + 1$
c	**i** $y = x^4$	**ii** $y = -x^4$	**iii** $y = -x^4 + 3$	**iv** $y = -(x + 3)^4$	**v** $y = -(x + 3)^4 - 1$
d	**i** $y = x^5$	**ii** $y = -2x^5$	**iii** $y = -2x^5 - 1$	**iv** $y = -(x - 1)^5$	**v** $y = -2(x - 1)^5$

EXAMPLE 1

Sketch the following functions. Show graphically and describe in words the relationship of each function to the graph of $y = x^2$.

a $y = 3x^2 - 1$ **b** $y = -\frac{1}{5}(x + 7)^2 + 4$

	Solve	Think	Apply
a		This is the graph of $y = x^2$ compressed horizontally and translated 1 unit down.	Begin with $y = x^2$ and apply each change to produce the new graph.
b		$y = -\frac{1}{5}(x + 7)^2 + 4$ $= -\left(\frac{1}{5}(x + 7)^2 - 4\right)$ This is the graph of $y = x^2$ stretched horizontally, translated 7 units to the left, translated vertically 4 units down and then reflected in the x-axis.	

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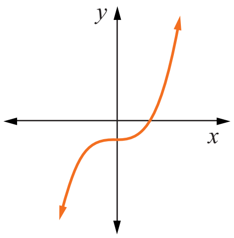
2 Sketch the following functions. Show graphically and describe the relationship of each function to $y = x^2$.

a $y = 5x^2$
b $y = \frac{3}{4}x^2$
c $y = x^2 - 8$
d $y = x^2 + 5$
e $y = (x - 4)^2$
f $y = 2(x - 4)^2$
g $y = (x + 7)^2$
h $y = \frac{1}{3}(x + 7)^2$
i $y = -5x^2$
j $y = -\frac{3}{4}x^2$
k $y = -(x^2 - 8)$
l $y = -x^2 - 5$
m $y = -(x - 4)^2$
n $y = -\frac{1}{3}(x + 6)^2$
o $y = (x - 1)^2 + 5$
p $y = -(x - 1)^2 - 5$
q $y = (x + 2)^2 - 1$
r $y = -(x + 2)^2 + 1$
s $y = 3(x - 6)^2 - 3$
t $y = -3(x - 6)^2 + 3$

3 Which of the following could be the graph of each function?

a $y = (x - 1)^3 - 5$

A


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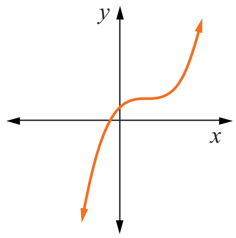

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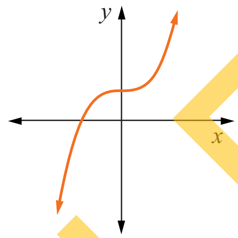

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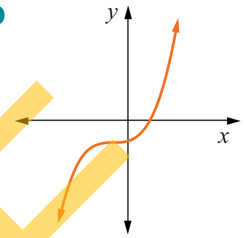
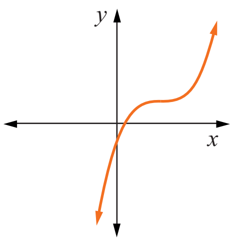


b $y = (x + 2)^3 + 4$

A


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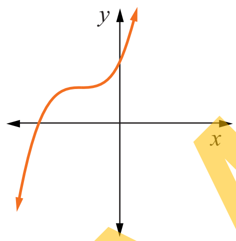
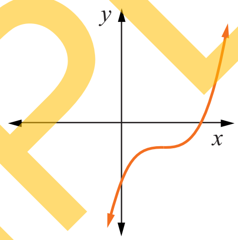


C


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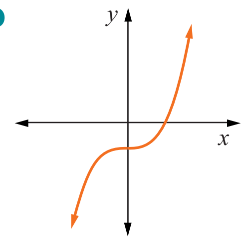
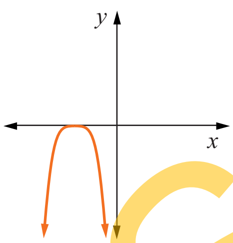


c $y = -(x - 3)^4$

A


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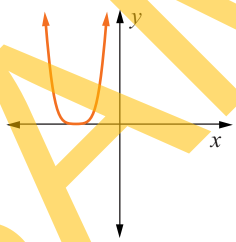

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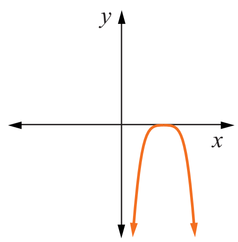
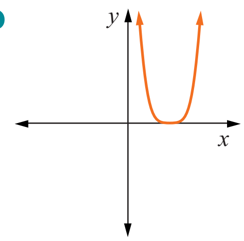
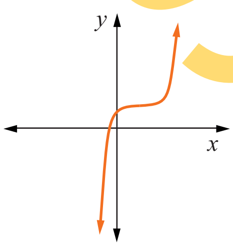


D


d $y = (x + 1)^5 - 2$

A


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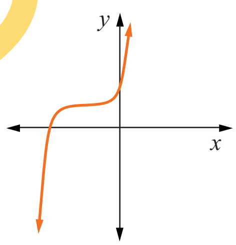

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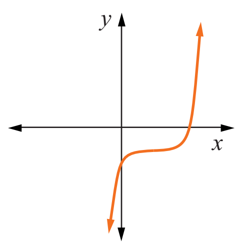
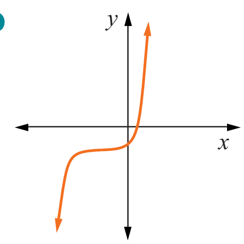
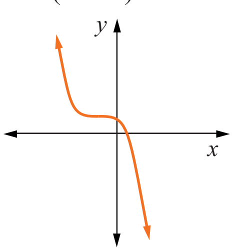
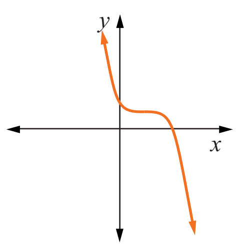


D


e $y = -(x + 1)^3 + 3$

A


B


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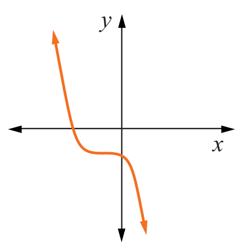
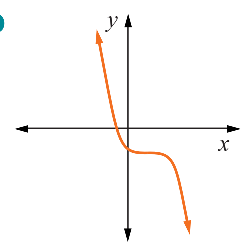


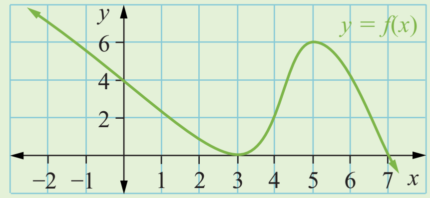
D


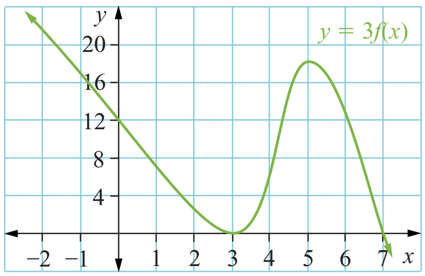
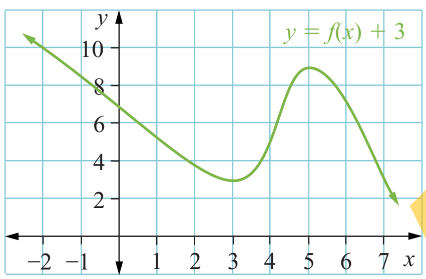
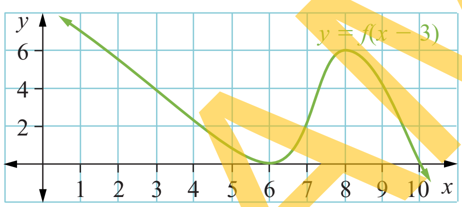
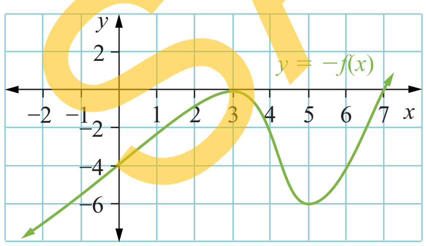
EXAMPLE 2

Use the graph of $y = f(x)$ given to sketch graphs of:

a $y = 3f(x)$ **b** $y = f(x) + 3$

c $y = f(x - 3)$ **d** $y = -f(x)$



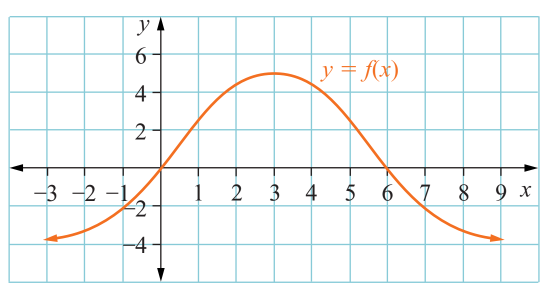
	Solve	Think	Apply
a		All points move three times further away from the x-axis.	Use the results from Investigations 1–5 to transform the original graph of $f(x)$ to produce the required graphs.
b		All points move up 3 units.	
c		All points move 3 units to the right.	
d		All points are reflected in the x-axis.	

4 Use the graph of $y = f(x)$ given and grid paper to sketch graphs of:

a $y = 2f(x)$ **b** $y = f(x) + 2$

c $y = f(x - 2)$ **d** $y = f(x + 2)$

e $y = -f(x)$

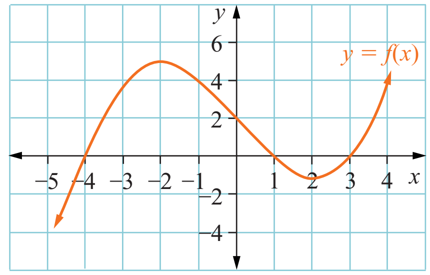


5 Use the graph of $y = f(x)$ given and grid paper to draw neat sketches of:

a $y = 3f(x)$ **b** $y = f(x) + 3$

c $y = f(x - 3)$ **d** $y = f(x + 3)$

e $y = -f(x)$



6 Without sketching, explain the similarities and differences between the curves $y = x^3 + x^2 + x$ and $y = x^3 + x^2 + x + 1$.

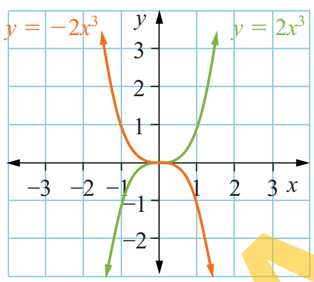
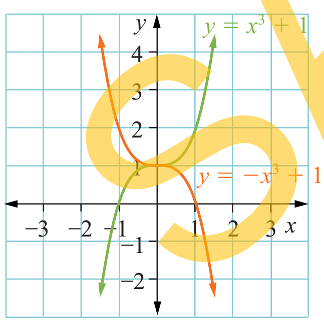
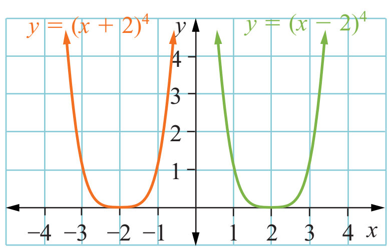
B Symmetry about the *y*-axis

EXAMPLE 1

a On the same diagram sketch the graphs of $y = f(x)$ and $y = f(-x)$ given:

i $f(x) = 2x^3$ **ii** $f(x) = x^3 + 1$ **iii** $f(x) = (x - 2)^4$

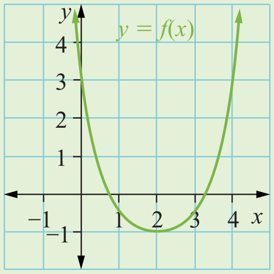
b Study the symmetry of the three graphs from part **a** and then describe in words the relationship between the graphs of $y = f(x)$ and $y = f(-x)$.

	Solve	Think	Apply
a i		$f(x) = 2x^3$ so $f(-x) = 2(-x)^3$ $= -2x^3$	The graph of $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis.
ii		$f(x) = x^3 + 1$ so $f(-x) = (-x)^3 + 1$ $= -x^3 + 1$	
iii		$f(x) = (x - 2)^4$ so $f(-x) = (-x - 2)^4$ $= (-1(x + 2))^4$ $= (-1)^4(x + 2)^4$ $= (x + 2)^4$	
b	The graphs $f(x)$ and $f(-x)$ are reflections of each other in the y-axis.		

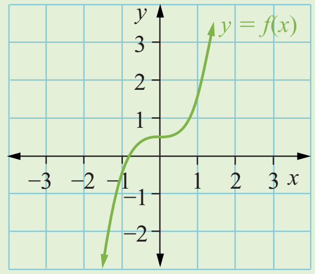
EXAMPLE 2

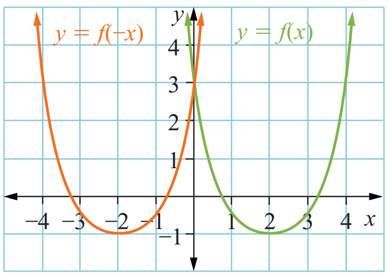
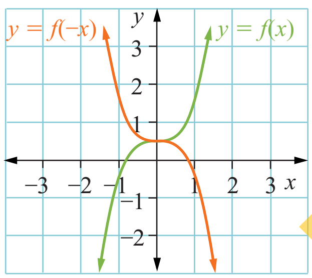
Given the graph of $y = f(x)$, sketch $y = f(-x)$.

a



b

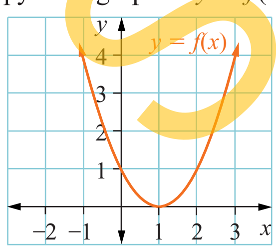


	Solve	Think	Apply
a		Reflect each curve in the y-axis.	The graphs $f(x)$ and $f(-x)$ intersect at the same point on the y-axis.
b			

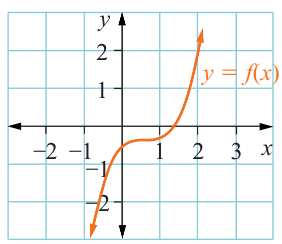
Exercise 15B

1 Copy each graph of $y = f(x)$ and sketch $y = f(-x)$ on the same axes.

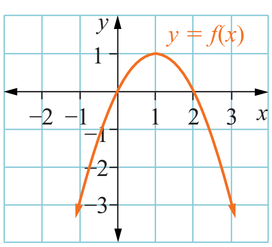
a



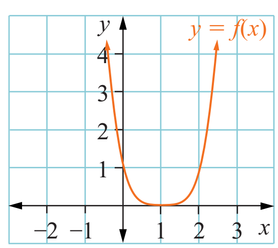
b



c



d



2 On the same diagram, sketch the graphs of $y = f(x)$ and $y = f(-x)$.

a $f(x) = -(x + 1)^2$ **b** $f(x) = -x^3 + 1$ **c** $f(x) = (x + 1)^3$ **d** $f(x) = (x - 1)^4$

EXAMPLE 3

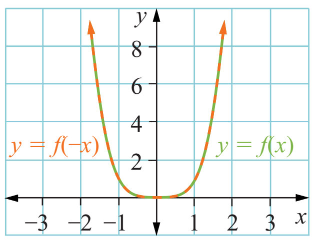
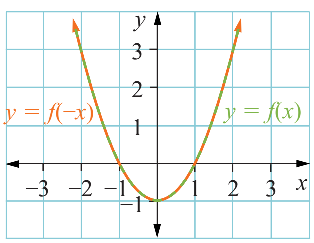
For each of the functions given:

i Sketch $y = f(x)$ and $y = f(-x)$.

ii Explain graphically what happens and why.

iii Determine $f(-x)$. What do you notice?

a $f(x) = x^4$ **b** $f(x) = x^2 - 1$

	Solve/Think	Apply
a i		If $f(-x) = f(x)$ then the graph of $y = f(x)$ is symmetrical about the y-axis.
ii	$y = f(x)$ is symmetrical about the y-axis, so when it is reflected in the y-axis it reflects onto itself.	
iii	$f(-x) = (-x)^4 = x^4$ $\therefore f(-x) = f(x)$	
b i		
ii	$y = f(x)$ is symmetrical about the y-axis, so when it is reflected in the y-axis it reflects onto itself.	
iii	$f(-x) = (-x)^2 - 1 = x^2 - 1$ $\therefore f(-x) = f(x)$	

EXAMPLE 4

Determine whether the graphs of the following functions are symmetrical about the y-axis.

a $f(x) = -7x^2$ **b** $f(x) = x^2 - 5x + 1$

	Solve	Think/Apply
a	$f(-x) = -7(-x)^2 = -7x^2$ $f(-x) = f(x)$ $\therefore y = f(x)$ is symmetrical about the y-axis.	Substitute $-x$ for x. If $f(-x) = f(x)$ then the graph is symmetrical about the y-axis. It is called an even function. An even polynomial function has all powers of x even.
b	$f(-x) = (-x)^2 - 5(-x) + 1 = x^2 + 5x + 1$ $f(-x) \neq f(x)$ $\therefore y = f(x)$ is not symmetrical about the y-axis.	

3 Determine whether the graphs of the following functions are symmetrical about the y-axis.

- **a** $f(x) = 5x^2$
- **b** $f(x) = 2x^3$
- **c** $f(x) = -6x^{10}$
- **d** $f(x) = x^2 + 7$
- **e** $f(x) = 3x^4 - 1$
- **f** $f(x) = x^3 + 1$
- **g** $f(x) = x^2 + x$
- **h** $f(x) = x^4 + x^2$
- **i** $f(x) = x^2 - 2x + 3$
- **j** $f(x) = 5x^6 - 7x^4 + 3x^2 + 2$

C Polynomials

A polynomial expression is one of the form:

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$$

where n is a non-negative integer (a positive integer or zero).

The **degree** of the polynomial is the highest power of x in the expression; that is, the value of n.

The **leading term** is the first term when the terms are written in descending powers of x.

The **leading coefficient** is the coefficient of the leading term. If the leading coefficient is 1 then the polynomial is said to be **monic**.

The **constant term** is the term containing x^0.

The function P defined by $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ is called a **polynomial function** of degree n.

EXAMPLE 1

Which of the following expressions are polynomials? Give reasons for your answers.

- **a** $5x^3 - 7x^2 + 2x + 1$
- **b** 25
- **c** $x^2 - 5x + \frac{3}{x} - 2$
- **d** $2x^4 - 6x^3 + 7\sqrt{x} + 1$

	Solve	Think	Apply
a	Yes	The powers of all terms in x are integers greater than or equal to zero, so it is a polynomial.	Polynomials must have positive integer powers.
b	Yes	$25 = 25x^0$; it is a polynomial of degree zero.	
c	No	It contains the term $\frac{3}{x} = 3x^{-1}$, which has a negative index.	
d	No	It contains a term $7\sqrt{x} = 7x^{\frac{1}{2}}$, which has a fractional index.	

Exercise 15C

1 State whether or not the following expressions are polynomials. Give reasons for your answers.

- **a** $7x^3 - 4x^2 + \frac{1}{2}$
- **b** $x^2 - \frac{2}{x^2} + 1$
- **c** $\frac{x^4 + x^2 - 1}{3}$
- **d** $\frac{3}{2x^3 - 4x^2 + 5x - 2}$
- **e** $-9\sqrt{x}$
- **f** $3^x - 2^x + 1$

EXAMPLE 2

For each of the following polynomials, state:

i the degree **ii** the leading term **iii** the leading coefficient

iv the constant term **v** whether the polynomial is monic.

a $15x^3 + 7x^2 - 8x - 3$ **b** $3x - 7x^2 + x^4$

	Solve	Think/Apply
a i	Degree = 3	The degree is the highest power. The leading term is the term with the highest power of x. A monic polynomial has the leading coefficient equal to 1.
ii	Leading term = $15x^3$	
iii	Leading coefficient = 15	
iv	Constant term = -3	
v	$15x^3 + 7x^2 - 8x - 3$ is not a monic polynomial.	
b i	Degree = 4	Rearrange the terms in descending powers of x, such as $x^4 - 7x^2 + 3x$ in part **b**. $3x - 7x^2 + x^4$ is a monic polynomial as the leading coefficient = 1.
ii	Leading term = x^4	
iii	Leading coefficient = 1	
iv	Constant term = 0	
v	$3x - 7x^2 + x^4$ is a monic polynomial.	

Note: 2, -3 and 11 are examples of polynomials of degree zero since $2 = 2x^0$, $-3 = -3x^0$, $11 = 11x^0$ and so on. These are sometimes called constant polynomials. However, the constant polynomial 0 (that is, a polynomial with all its coefficients zero) is called the zero polynomial and is considered to have no degree.

2 For each of the following polynomials state:

i the degree **ii** the leading term **iii** the leading coefficient

iv the constant term **v** whether the polynomials is monic.

a $3x^5 - 2x^3 + 7x + 11$ **b** $-7 - 5x + x^2$ **c** $\frac{1}{3}x^3 + \frac{1}{2}x^2 + x - 1$

d $4x - 2x^3 + 5x^2 - 1$ **e** $3x + 2$ **f** 5

EXAMPLE 3

If $P(x) = x^3 - 2x^2 + 5x + 1$, find the value of $P(-2)$ and $P(0)$.

Solve	Think	Apply
$P(-2) = (-2)^3 - 2(-2)^2 + 5(-2) + 1$ $= -25$ $P(0) = 0^3 - 2 \times 0^2 + 5 \times 0 + 1$ $= 1$	$P(-2)$ is the value of the polynomial when $x = -2$. Substitute $x = -2$. Substitute $x = 0$.	Substitute the value of x and calculate.

3 For each polynomial function find the values indicated.

a $P(x) = 3x^2 - 7x + 1$ find $P(0)$, $P(1)$, $P(-2)$

b $Q(x) = 25x - 27$ find $Q(-1)$, $Q(1)$, $Q(0)$

c $R(x) = 3x^6$ find $R(2)$, $R(-3)$, $R(0)$

4 For each polynomial given evaluate $P(10)$.

a $P(x) = 7x + 3$
b $P(x) = 8x^2 + 5x + 6$
c $P(x) = 2x^3 + 4x^2 + 6x + 1$
d $P(x) = 3x^4 + 2x^3 + 4x^2 + 5x + 9$
e $P(x) = 9x^5 + 7x^4 + 6x^3 + 5x^2 + 8x + 4$
f $P(x) = 5x^3 + 2x^2 + 3$
g $P(x) = x^4 + 3x^2$
h $P(x) = x^3 + x + 1$

5 **a** **i** Evaluate $P(x) = 2x - 1$ for $x = 1, 2, 3, \ldots$; that is, find $P(1), P(2), P(3), \ldots$

ii Which well known sequence of numbers is formed?

b Evaluate $Q(x) = x^2$ for $x = 1, 2, 3, \ldots$

c Evaluate $H(x) = 2x^3$ for $x = 1, 2, 3, \ldots$

6 Can you find a polynomial $P(x)$ that would generate the following number patterns for $x = 1, 2, 3, 4, \ldots$?

a $2, 4, 6, 8, \ldots$
b $2, 5, 10, 17, \ldots$
c $1, 8, 27, 64, \ldots$
d $2, 5, 8, 11, \ldots$
e $5, 8, 11, 14, \ldots$
f $1, 4, 9, 16, \ldots$

7 Find the polynomial function $P(x)$ given the following information.

a Degree = 1, leading coefficient = 3, constant term = 5
b Degree = 1, leading coefficient = 2, $P(1) = 3$
c Degree = 2, monic, constant term = 1, $P(2) = 13$
d Leading term = $4x^3$, constant term = 1, $P(1) = 9$, $P(-1) = -5$
e Degree = 3, leading coefficient = 2, $P(0) = 1$, $P(1) = 0$, $P(2) = 1$

D The algebra of polynomials

Polynomials can be added, subtracted and multiplied using the laws of algebra learnt in previous work.

Addition and subtraction of polynomials

EXAMPLE 1

a Find the sum of the polynomials $P(x) = 4x^3 - 3x^2 + 7x - 1$ and $Q(x) = 2x^3 + 6x^2 - 2x - 5$.
b Find the difference $P(x) - Q(x)$.

	Solve	Think	Apply
a	$P(x) + Q(x)$ $= (4x^3 - 3x^2 + 7x - 1) + (2x^3 + 6x^2 - 2x - 5)$ $= (4x^3 + 2x^3) + (-3x^2 + 6x^2) + (7x - 2x) + (-1 - 5)$ $= 6x^3 + 3x^2 + 5x - 6$	The working could be set out as: $4x^3 - 3x^2 + 7x - 1$ $+\ 2x^3 + 6x^2 - 2x - 5$ $6x^3 + 3x^2 + 5x - 6$	Writing one polynomial under the other makes calculation easier. Ensure terms with the same power are under each other.
b	$P(x) - Q(x)$ $= (4x^3 - 3x^2 + 7x - 1) - (2x^3 + 6x^2 - 2x - 5)$ $= 4x^3 - 3x^2 + 7x - 1 - 2x^3 - 6x^2 + 2x + 5$ $= 2x^3 - 9x^2 + 9x + 4$	Also: $4x^3 - 3x^2 + 7x - 1$ $-\ 2x^3 + 6x^2 - 2x - 5$ $2x^3 - 9x^2 + 9x + 4$	

If $P(x)$ and $Q(x)$ are polynomial expressions then:

the *sum* $P(x) + Q(x)$ and the *difference* $P(x) - Q(x)$ are also polynomial expressions.

Exercise 15D

1 **a** Find the sum $P(x) + Q(x)$ and the difference $P(x) - Q(x)$ of the polynomials given.

i $P(x) = 3x^2 - 5x + 2$, $Q(x) = x^2 + 7x - 3$

ii $P(x) = x^3 - 2x^2 + 3x + 7$, $Q(x) = 2x^3 + 3x^2 - 5x + 2$

iii $P(x) = 2x^3 + 9x^2 + 8x + 1$, $Q(x) = 6x^2 - 5x - 3$

iv $P(x) = 3x^4 + 2x^3 - 3x^2 + 4x - 2$, $Q(x) = 2x^4 + 3x^3 - 7x^2 + 5x - 11$

v $P(x) = x^4 + 9x^3 + 7x^2 - 4x + 9$, $Q(x) = 4x^3 - 8x^2 - 15$

b How is the degree of $P(x) \pm Q(x)$ related to the degree of $P(x)$ and $Q(x)$ for these polynomials?

c Will the result of part **b** be true for all polynomials $P(x)$ and $Q(x)$? Discuss. (For example, consider what happens if the leading terms of $P(x)$ and $Q(x)$ are opposites.)

2 Given $P(x) = x^3 - 3x^2 + 10x - 11$, $Q(x) = x^3 + 8x^2 - 7x + 3$ and $R(x) = 12x^2 + 15x - 1$ find:

a $P(x) + Q(x) - R(x)$

b $P(x) - Q(x) + R(x)$

c $P(x) - Q(x) - R(x)$

d $-P(x) + Q(x) - R(x)$

e $-P(x) - Q(x) - R(x)$

f $P(x) + Q(x) + R(x)$

Multiplication of polynomials

EXAMPLE 2

Given $P(x) = 3x - 2$ and $Q(x) = x^3 + 7x^2 - 5x + 8$, find $P(x) \times Q(x)$.

Solve	Think	Apply
$P(x) \times Q(x)$ $= (3x - 2)(x^3 + 7x^2 - 5x + 8)$ $= 3x(x^3 + 7x^2 - 5x + 8)$ $\quad - 2(x^3 + 7x^2 - 5x + 8)$ $= 3x^4 + 21x^3 - 15x^2 + 24x$ $\quad - 2x^3 - 14x^2 + 10x - 16$ $= 3x^4 + 19x^3 - 29x^2 + 34x - 16$	The working can be set out as follows: $x^3 + 7x^2 - 5x + 8$ $\times \quad 3x - 2$ $-2x^3 - 14x^2 + 10x - 16$ $+ \quad 3x^4 + 21x^3 - 15x^2 + 24x$ $3x^4 + 19x^3 - 29x^2 + 34x - 16$	If $P(x)$ and $Q(x)$ are polynomials, then the product $P(x) \times Q(x)$ is also a polynomial.

3 Find the product of the polynomials $P(x)$ and $Q(x)$.

a $P(x) = x + 5$, $Q(x) = x^2 - 3x + 7$

b $P(x) = 2x + 1$, $Q(x) = x^3 + x^2 + 2x + 3$

c $P(x) = 1 - 5x$, $Q(x) = 7x^2 + 21x - 3$

d $P(x) = 4x + 3$, $Q(x) = 2x^3 + 5x^2 - 7x + 1$

e $P(x) = 1 - 6x$, $Q(x) = x^3 - 2x^2 + 4x - 9$

4 Describe in words how the degree of $P(x) \times Q(x)$ is related to the degree of $P(x)$ and $Q(x)$.

5 Find the following products.

a $(x^2 + x + 1)(x^3 + 2x^2 - 3x - 5)$

b $(2x^2 - 3x + 4)(3x^3 + 4x^2 - 1)$

c $(x^2 + 3x + 2)^2$

d $(x - 3)(2x + 1)(x^2 - 3x - 4)$

e $(x + 5)(x - 2)(3x + 1)(4x - 3)$

f $(x - 2)(x - 5)(2x + 1)(3x - 1)$

E Division of polynomials

Before undertaking the division of polynomials, it is best to consider the method for long division of integers.

EXAMPLE 1

Divide 385 by 18.

Solve	Think	Apply
2 1 18) 3 8 5 − 3 6 2 5 − 1 8 7 385 ÷ 18 = 21 r 7	The result of the long division algorithm may be expressed as 385 ÷ 18 = 21 with a remainder of 7; that is, 385 ÷ 18 = 21 r 7 or 385 = 18 × 21 + 7	In the long division process, dividend = divisor × quotient + remainder: 385 is called the dividend. 18 is called the divisor. 21 is called the quotient. 7 is called the remainder.

EXAMPLE 2

Divide 17 368 by 34 and write the result in the form 17 368 = 34 × quotient + remainder.

Solve	Think	Apply
5 1 0 34) 1 7 3 6 8 − 1 7 0 3 6 − 3 4 2 8 17 368 = 34 × 510 + 28	So 17 368 ÷ 34 = 510 r 28 or 17 368 = 34 × 510 + 28.	The division process continues until the dividend is less than the divisor. The number left is the remainder.

EXAMPLE 3

Use the long division algorithm from Examples 1 and 2 to divide the polynomial $3x^2 - 11x + 2$ by $x - 1$. Express the result in the form dividend = divisor × quotient + remainder.

Solve	Think	Apply
$3x - 8$ $x - 1\overline{)\,3x^2 - 11x + 2}$ $-\ (3x^2 - 3x)$ $-8x + 2$ $-\ (-8x + 8)$ -6 $3x^2 - 11x + 2$ $= (x - 1)(3x - 8) + (-6)$	Divide x into $3x^2$. $3x \times (x - 1)$ Subtract. Bring down the 2. Divide x into $-8x$. $-8 \times (x - 1)$ Subtract. Express the result as dividend = divisor × quotient + remainder.	Each term in the quotient is obtained by dividing the leading term of the divisor (x) into the leading term of the dividend, first $3x^2$ and then $-8x$ in this example. *Note:* Degree of remainder < degree of divisor.

EXAMPLE 4

Divide $8x^3 + 4x^2 + 12x - 5$ by $2x + 1$ and express the result in this form:
$8x^3 + 4x^2 + 12x - 5 = (2x + 1) \times Q(x) + R$

Solve	Think	Apply
$\begin{array}{r} 4x^2 \qquad + 6 \\ 2x + 1 \overline{) 8x^3 + 4x^2 + 12x - 5} \\ -(8x^3 + 4x^2) \qquad\qquad \\ \hline 12x - 5 \\ -\quad (12x + 6) \\ \hline -11 \end{array}$ $8x^3 + 4x^2 + 12x - 5$ $= (2x + 1)(4x^2 + 6) + (-11)$	Divide $2x$ into $8x^2$. Subtract. Bring down $12x - 5$. Divide $2x$ into $12x$. Subtract. *Note:* Degree of remainder $<$ degree of divisor.	1 If we use the long division algorithm to divide one polynomial $P(x)$ by another $D(x)$, we get a unique quotient $Q(x)$ and a unique remainder $R(x)$ such that $P(x) = D(x) \times Q(x) + R(x)$ where the degree $R(x) <$ degree $D(x)$. 2 If $D(x)$ is degree 1 (linear polynomial), degree $R(x) < 1$. Degree $R(x) = 0$ or $R(x) =$ a constant $= R$.

Exercise 15E

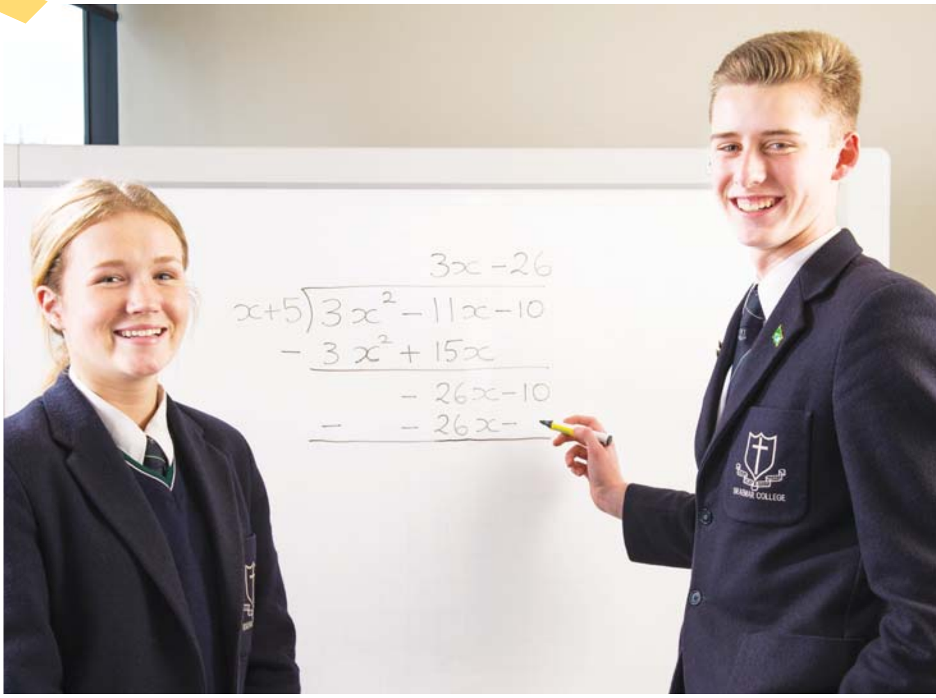
1 Complete the following.

a
$$\begin{array}{r} 2\square 7 \\ 15 \overline{) 3\ 8\ 6\ 4} \\ -\ 3\ 0\quad \\ \hline 8\square\ \\ -\quad 7\ 5\ \\ \hline 1\ 1\ 4 \\ -\quad \square\square\square \\ \hline 9 \end{array}$$

b
$$\begin{array}{r} 2x^2 - \square + 2 \\ 3x - 2 \overline{) 6x^3 - 7x^2 + 8x - 5} \\ -(6x^3 - 4x^2) \qquad\qquad \\ \hline \square + 8x \quad \\ -\quad (-3x^2 + 2x) \quad \\ \hline 6x - \square \\ -\quad (6x - 4) \\ \hline \square \end{array}$$

2 Perform the following divisions and express each result in this form:
dividend = divisor × quotient + remainder

a 12 564 ÷ 28
b 92 156 ÷ 18
c $(3x^2 - 11x - 10) \div (x + 5)$
d $(-2x^2 + 10x - 3) \div (x + 1)$
e $(x^3 + 2x^2 - 8x - 6) \div (x - 2)$
f $(3x^3 - 4x^2 + 3x + 2) \div (x + 3)$
g $(x^4 + 4x^3 - 3x^2 + 2x - 1) \div (x - 1)$
h $(2x^3 + x^2 - 4x - 7) \div (2x - 3)$
i $(3x^3 + 4x^2 - x + 7) \div (3x - 2)$
j $(2x^4 - 3x^3 + 6x^2 + 2x + 5) \div (2x + 1)$



EXAMPLE 5

By dividing prove that $x - 2$ is a factor of $2x^3 + x^2 - 8x - 4$.
Hence express $2x^3 + x^2 - 8x - 4$ as the product of three linear factors.

Solve	Think/Apply
$$\begin{array}{r} 2x^2 + 5x + 2 \\ x - 2\,\overline{)\,2x^3 + x^2 - 8x - 4} \\ -\ (2x^3 - 4x^2) \\ \hline 5x^2 - 8x \\ -\ (-5x^2 - 10x) \\ \hline 2x - 4 \\ -\ (2x - 4) \\ \hline 0 \end{array}$$	As the remainder is zero, $(x - 2)$ is a factor of $2x^3 + x^2 - 8x - 4$.
$2x^3 + x^2 - 8x - 4 = (x - 2)(2x^2 + 5x + 2)$ $= (x - 2)(2x + 1)(x + 2)$	Factorise the quadratic.

3 For each of the following, show by division that the first polynomial is a factor of the second polynomial.

a $x + 2,\ x^3 + 5x^2 + 5x - 2$
b $2x - 1,\ 2x^3 - x^2 + 2x - 1$
c $x + 1,\ x^3 + 3x^2 + 3x + 1$
d $3x + 4,\ 3x^3 + 10x + 11x + 4$
e $x - 2,\ x^4 + x^3 - 8x^2 + 5x - 2$
f $x + 2,\ x^4 - x^3 - 9x^2 + 3x + 18$

4 Express each polynomial $P(x)$ as the product of three linear factors given that $h(x)$ is one of these factors.

a $P(x) = x^3 + 4x^2 + x - 6,\ h(x) = x - 1$
b $P(x) = 2x^3 - 5x^2 - x + 6,\ h(x) = 2x - 3$
c $P(x) = 2x^3 - 15x^2 + 22x + 15,\ h(x) = x - 5$
d $P(x) = 4x^3 + 12x^2 - x - 3,\ h(x) = x + 3$
e $P(x) = 8x^3 + 36x^2 + 54x + 27,\ h(x) = 2x + 3$
f $P(x) = 8x^3 - 12x^2 + 6x - 1,\ h(x) = 2x - 1$

5 Find the quotient and remainder for each of the following divisions.

a $(3x^3 - 2x^2 - 27x - 18) \div (3x - 2)$
b $(-6x^3 + x^2 + 4x + 3) \div (2x + 1)$
c $(5x^3 + 7x - 4) \div (x - 3)$
d $(x^3 - 1) \div (x + 1)$
e $(6x^3 - 5x^2 - 2) \div (2x - 1)$
f $(x^4 - a^4) \div (x - a)$
g $(4x^3 - 7x^2 + 2x - 3) \div (x^2 + 2x - 1)$
h $(2x^3 + 5x^2 + 11x - 1) \div (x^2 - x + 3)$

F The remainder theorem

When $2x^3 - 5x^2 + 8x - 10$ is divided by $x - 3$ we get a quotient of $2x^2 + x + 11$ and a remainder of 23; that is, if $P(x) = 2x^3 - 5x^2 + 8x - 10$ then $P(x) = (x - 3)(2x^2 + x + 11) + 23$.

If we substitute $x = 3$ into this statement then:

$$\begin{aligned} P(3) &= (3 - 3)(2(3)^2 + (3) + 11) + 23 \\ &= 0 \times 32 + 23 \\ &= 23 \end{aligned}$$

So $P(3)$ is the remainder when $P(x)$ is divided by $(x - 3)$.

This is a particular example of a general result known as the **remainder theorem**.

The remainder theorem states that:

> If the polynomial $P(x)$ is divided by $(x - a)$ until a constant remainder is obtained, then the remainder is $P(a)$.

Proof: Let $Q(x)$ be the quotient and $R(x)$ the remainder when $P(x)$ is divided by $(x - a)$.

Note: Degree $R(x) <$ degree $(x - a)$, so degree $R(x) < 1$ and $R(x)$ must be a constant.

$$P(x) = (x - a)Q(x) + R$$

$$P(a) = (a - a)Q(a) + R \qquad \text{Substitute } x = a.$$

$$= 0 \times Q(a) + R$$

$$= R$$

Thus the remainder $R = P(a)$.

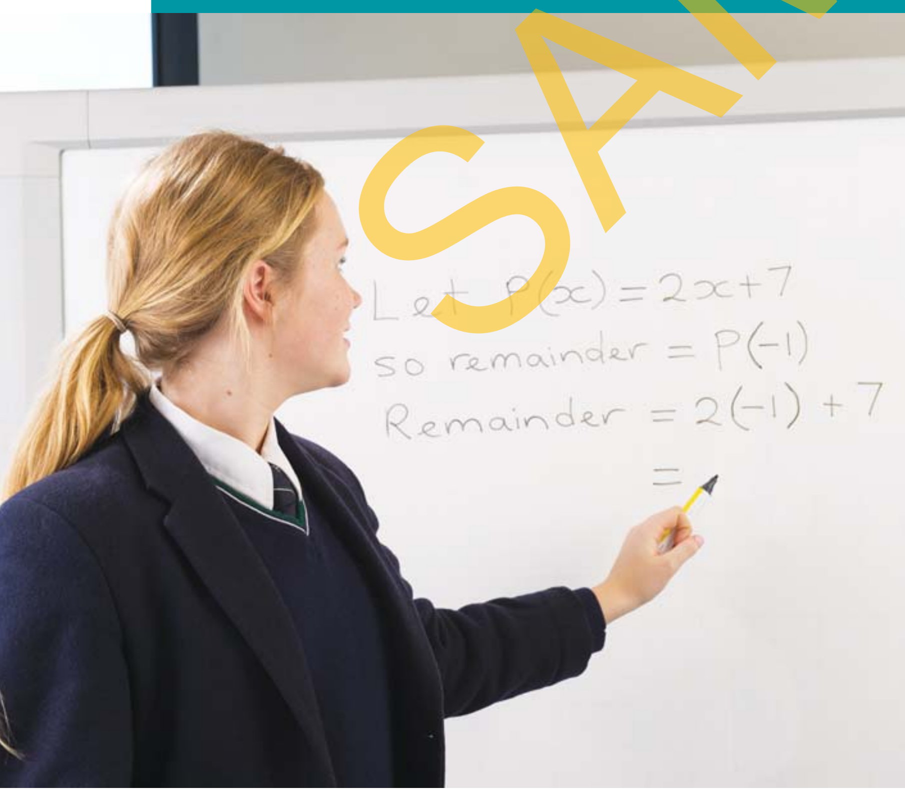
EXAMPLE 1

Find the remainder when $2x^3 - 5x^2 + 7x - 4$ is divided by the following:

a $x - 2$ **b** $x + 2$

	Solve	Think	Apply
a	Let $P(x) = 2x^3 - 5x^2 + 7x - 4$, so remainder $= P(2)$. Remainder $= 2(2)^3 - 5(2)^2 + 7(2) - 4$ $= 6$	Substitute $x = 2$ and evaluate.	To find the remainder when $P(x)$ is divided by $(x - a)$ evaluate $P(a)$.
b	Let $P(x) = x + 2 = x - (-2)$, so the remainder is $P(-2)$. Remainder $= 2(-2)^3 - 5(-2)^2 + 7(-2) - 4$ $= -54$	Substitute $x = -2$ and evaluate.	

Exercise 15F



1 Use the remainder theorem to find the remainder when the first polynomial is divided by the second.

a $2x + 7, x + 1$

b $3x^2 + 12x - 28, x - 1$

c $5x^2 - 11x - 31, x - 4$

d $x^3 - 4x^2 + 3x + 1, x + 1$

e $-x^3 + 2x^2 - 7x - 13, x + 5$

f $2x^3 + 7x^2 - 10x - 5, x - 3$

g $-4x^3 - x^2 + x + 1, x + 2$

h $x^4 + x^3 - 2x^2 - 4x + 3, x + 2$

i $3x^4 - 5x^3 - 6x^2 + 8x - 6, x - 2$

j $2x^4 - 25x^3 + 71x^2 - 56x + 35, x$

2 **a** When $P(x) = ax^2 - 8x + 3$ is divided by $x - 7$ the remainder is 45. Find a.

b When $F(x) = x^3 + 5x^2 + kx + 1$ is divided by $x + 5$ the remainder is 31. Find k.

c When $Q(x) = ax^3 + bx^2 + x - 1$ is divided by $(x - 1)$ the remainder is -3, and when divided by $(x + 2)$ the remainder is -39. Find a and b.

3 Find the remainder when $P(x)$ is divided by $g(x)$.

a $P(x) = x^3 + 5x^2 + 3x - 6$, $g(x) = x + 2$

b $P(x) = x^3 - 2x^2 - 5x + 6$, $g(x) = x - 3$

c Comment on the meaning of the results in parts **a** and **b**.

G The factor theorem

If, when $P(x)$ is divided by $(x - a)$, the remainder is zero, then $(x - a)$ is a factor of $P(x)$.
So if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

EXAMPLE 1

Show that $x - 3$ is a factor of $x^3 - 4x^2 + x + 6$.

Solve	Think	Apply
Let $P(x) = x^3 - 4x^2 + x + 6$ then $P(3) = 27 - 36 + 3 + 6 = 0$ $\therefore x - 3$ is a factor of $P(x)$.	Substitute $x = 3$ into $P(x)$.	If $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

Exercise 15G

1 Verify that the first polynomial is a factor of the second.

a $x - 1$, $3x^3 + 5x^2 - 4x - 4$

b $x + 1$, $4x^3 + 3x^2 + 9x + 10$

c $x - 2$, $2x^3 + 3x^2 - 18x + 8$

d $x + 2$, $2x^4 + 2x^3 + x^2 + x - 2$

e $x + 4$, $2x^3 + 3x^2 - 12x + 32$

f $x - 3$, $3x^4 - 9x^3 + 5x^2 - 7x - 24$

EXAMPLE 2

Find all the linear factors of the polynomial $P(x) = x^3 + 6x^2 + 11x + 6$.

Solve	Think	Apply
$\begin{array}{r} x^2 + 5x + 6 \\ x + 1 \overline{) x^3 + 6x^2 + 11x + 6} \\ -(x^3 + x^2) \\ \hline 5x^2 + 11x \\ -(5x^2 + 5x) \\ \hline 6x + 6 \\ -(6x + 6) \\ \hline 0 \end{array}$ $P(x) = (x + 1)(x + 2)(x + 3)$	The factors of the constant term 6 are ±1, ±2, ±3 and ±6, so we try $P(\pm1)$, $P(\pm2)$, $P(\pm3)$ and $P(\pm6)$ until we find a factor. $P(1) = 1 + 6 + 11 + 6 \neq 0$ $\therefore x - 1$ is not a factor. $P(-1) = -1 + 6 - 11 + 6 = 0$ $\therefore x + 1$ is a factor. We can find the other factors by factorising the quotient: $(x^2 + 5x + 6) = (x + 2)(x + 3)$	Once a single linear factor of a cubic is found then polynomial division gives a quadratic that can be factorised.

2 Use the factor theorem to find all the linear factors of the following polynomials.

a $x^3 + 3x^2 - 6x - 8$
b $x^3 - 7x + 6$
c $x^3 + 3x^2 - 16x + 12$
d $x^3 + 6x^2 + 12x + 8$
e $x^3 + 6x^2 - x - 6$
f $x^3 + 3x^2 - 10x - 24$
g $x^3 - 7x^2 + 36$
h $x^4 + x^3 - 3x^2 - 4x - 4$
i $x^4 + 2x^3 - 3x^2 - 4x + 4$
j $2x^3 - 13x^2 - 13x + 42$

3 **a** If $x - 3$ is a factor of $6x^3 + kx^2 + 2x + 3$, find the value of k.
b If $P(x) = x^3 + ax^2 + bx - 6$ is divisible by $(x + 1)$ and $(x - 2)$, find the values of a and b.
c $(x + 1)$ is a factor of $f(x) = x^3 + mx^2 + nx - 3$, and when $f(x)$ is divided by $(x + 3)$ the remainder is -24. Find m and n.
d Find the values of a and b if $(x - 1)$ is a factor of $g(x) = ax^4 - 5x^3 + b$, and when $g(x)$ is divided by $(x + 2)$ the remainder is 135.
e Find the values of k and m if $P(x) = x^3 + kx^2 - 12x + m$ is divisible by $x^2 - x - 6$.

4 **a** If n is a positive integer, show that $(x - a)$ is always a factor of $x^n - a^n$.
b Is $(x + a)$ always a factor of $x^n + a^n$? Discuss and explain.

H Polynomial equations

Previously we have solved polynomial equations of degree 1 (linear equations) and of degree 2 (quadratic equations). Polynomial equations of degree 3 or higher can be difficult to solve, but in some cases we can use the factor theorem to help us.

EXAMPLE 1

Solve $x^3 + 5x^2 - 12x - 36 = 0$.

Solve	Think	Apply
$x^2 + 3x - 18$ $x + 2\,\overline{)\,x^3 + 5x^2 - 12x - 36}$ $-\,(x^3 + 2x^2)$ $3x^2 - 12x$ $-\,(3x^2 + 6x)$ $-18x - 36$ $-\,(-18x - 36)$ 0 $P(x) = (x + 2)(x^2 + 3x - 18)$ $= (x + 2)(x - 3)(x + 6)$ Thus $x = -2, 3, -6$	Let $P(x) = x^3 + 5x^2 - 12x - 36$. To find the factors of $P(x)$ try $P(\pm 1)$, $P(\pm 2)$, and so on. $P(1) = 1 + 5 - 12 - 36 \neq 0$ $P(-1) = -1 + 5 + 12 - 36 \neq 0$ $P(2) = 8 + 20 - 24 - 36 \neq 0$ $P(-2) = -8 + 20 + 24 - 36$ $= 0$ $\therefore (x + 2)$ is a factor of $P(x)$. We can find the other factors by factorising the quotient: $x^2 + 3x - 18 = (x - 3)(x + 6)$	Any value of x that makes the value of $P(x)$ equal to zero is called a zero of the polynomial. The zeros of the polynomial $P(x) = x^3 + 5x^2 - 12x - 36$ are -2, 3 and -6.

EXAMPLE 2

Solve $x^3 - 2x^2 - 2x - 3 = 0$.

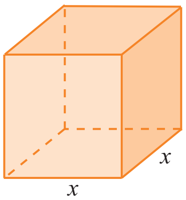
Solve	Think	Apply
$$\begin{array}{r} x^2 + x + 1 \\ x - 3\overline{)\,x^3 - 2x^2 - 2x - 36} \\ -\,(x^3 - 3x^2) \\ \hline x^2 - 2x \\ -\quad (x^2 - 3x) \\ \hline x - 3 \\ -\quad (-x - 3) \\ \hline 0 \end{array}$$ $x = 3$ or $x^2 + x + 1 = 0$ $\therefore x = 3$ is the only solution.	Let $f(x) = x^3 - 2x^2 - 2x - 3$ $f(1) = 1 - 2 - 2 - 3 \neq 0$ $f(-1) = -1 - 2 + 2 - 3 \neq 0$ $f(3) = 27 - 18 - 6 - 3 = 0$ $\therefore (x - 3)$ is a factor of $f(x)$. $x^3 - 2x^2 - 2x - 3 = 0$ $\therefore (x - 3)(x^2 + x + 1) = 0$ If the quadratic does not factorise, use the quadratic formula to solve. $x = \frac{-1 \pm \sqrt{1 - 4}}{2}$ has no solutions.	If the polynomial $f(x) = x^3 - 2x^2 - 2x - 3$ has only one zero (3), it follows that the zeros of the polynomial $P(x)$ are the solutions of the polynomial equation $P(x) = 0$. A quadratic equation has no real roots if the discriminant $\Delta = b^2 - 4ac < 0$. In this case $\Delta = -3$.

Exercise 15H

1 Solve the following polynomial equations.

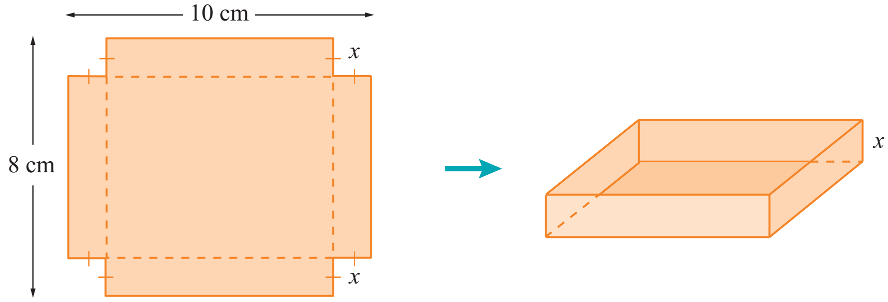
a $x^3 - 3x^2 - 10x + 24 = 0$
b $x^3 + 7x^2 + 7x - 15 = 0$
c $x^3 + 10x^2 = 24x$
d $x^3 + 9x^2 + 26x + 24 = 0$
e $2x^3 - 3x^2 = 11x - 6$
f $x^3 = 3x^2 - 3x + 1$
g $x^3 + 6x^2 + 7x + 2 = 0$
h $x^3 + 2x^2 - 6x = 4$
i $x^4 + x^3 - 3x^2 - 4x - 4 = 0$
j $x^4 + 2x^3 - 4x^2 - 7x - 2 = 0$

2 A box is to be built with a square base and height 2 cm more than the length of the base.



a Write an expression for the volume of the box.
b If the volume is to be 45 cm^3, find a polynomial equation from which the dimensions of the box can be found.
c Find the dimensions that will give this volume.

3 Four identical squares are cut from the corners of a rectangular sheet of metal 10 cm × 8 cm. The sides are folded along the dotted lines to form a rectangular box, as shown.



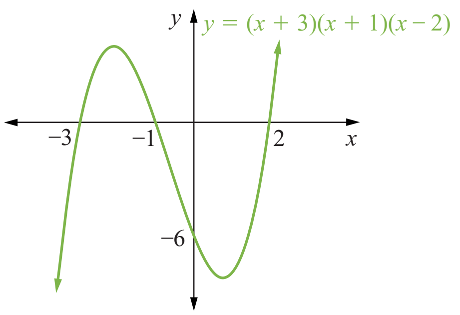
a Find an expression for the volume of the box.
b Find the side length of the square that should be cut from each corner if the volume is to be 48 cm^3.
c Hence find the dimensions of the rectangular box with this volume.

I Sketching polynomials

We sketched some simple polynomials in Section 15A. We now extend this to more general examples.

EXAMPLE 1

Find the zeros of the polynomial function $y = (x + 3)(x + 1)(x - 2)$ and hence sketch its graph.

Solve	Think	Apply
	Let $y = 0$: $x = -3, -1$ and 2 are the x-intercepts. Let $x = 0$: $y = -6$ is the y-intercept. As $x \to +\infty, y \to +\infty$ $x \to -\infty, y \to -\infty$	If the leading term is positive, then a cubic graph looks similar to $y = x^3$ as $x \to \pm\infty$.

Exercise 15I

1 Sketch these polynomials.

a $y = (x + 5)(x + 2)(x + 1)$

b $y = -2(x + 3)(x - 2)(x - 3)$

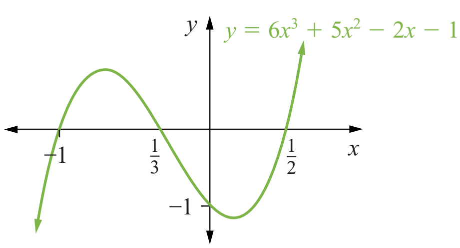
c $y = x(x + 4)(x - 3)$

d $y = x(x - 2)(x - 5)(x + 1)$

e $y = (4 - x)(x - 3)(x - 1)(x + 2)$

EXAMPLE 2

Find the zeros of the polynomial function $P(x) = 6x^3 + 5x^2 - 2x - 1$. Hence sketch the graph of $y = P(x)$.

Solve	Think/Apply
$6x^2 - x - 1$ $x + 1\,\overline{)\,6x^3 + 5x^2 - 2x - 1}$ $-\,(6x^3 + 6x^2)$ $-x^2 - 2x$ $-\,(-x^2 - x)$ $-x - 1$ $-\,(-x - 1)$ 0 	$P(x) = 6x^3 + 5x^2 - 2x - 1$ $P(1) = 6 + 5 - 2 - 1 \neq 0$ $P(-1) = -6 + 5 + 2 - 1 = 0$ $\therefore (x + 1)$ is a factor. $P(x) = (x + 1)(6x^2 - x - 1)$ $= (x + 1)(2x - 1)(3x + 1)$ The zeros of the polynomial are $-1, \frac{1}{2}, -\frac{1}{3}$. $\therefore$ The x-intercepts are $-1, \frac{1}{2}, -\frac{1}{3}$. Let $x = 0$: $y = -1$ *Note:* This is the constant term of $P(x)$. As $x \to +\infty, y \to +\infty$ $x \to -\infty, y \to -\infty$ We put all this information together to sketch the graph.

Notes:

1 The graphs of polynomial functions are continuous curves; that is, they have no gaps in them, as there is a value of y for every value of x.

2 For very large positive values of x (as $x \to +\infty$) and for very large negative values of x (as $x \to -\infty$) the value of y approximately equals the value of the leading term. Hence the sign of the leading term of a polynomial determines whether $y \to \pm\infty$ as $x \to \pm\infty$.

3 In Examples 1 and 2, y is a polynomial function of degree 3. A polynomial of degree 3 can have no more than three linear factors, therefore it can have no more than three zeros. In general, a polynomial of degree n can have no more than n zeros.

2 Use the factor theorem to find the zeros of the following polynomials and hence sketch each function.

a $P(x) = x^3 + 2x^2 - x - 2$

b $P(x) = x^3 + 2x^2 - 5x - 6$

c $P(x) = x^3 + 3x^2 - 70x$

d $P(x) = x^3 + 9x^2 + 26x + 24$

e $P(x) = -x^3 + 5x^2 + 4x - 20$

f $P(x) = x^4 - 19x^2 + 30x$

g $P(x) = -x^4 - 3x^3 - x^2 + 3x + 2$

h $P(x) = x^4 - 2x^3 - 15x^2$

J Significance of double and triple roots

EXAMPLE 1

a Give an example of a polynomial equation of degree 2 with:

i two distinct (different roots)

ii one distinct root

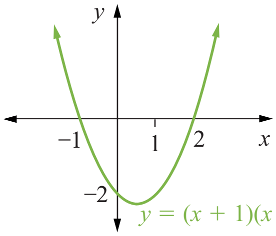
iii no roots.

b Sketch the graph of each of the corresponding polynomial functions.

	Solve	Think/Apply
a i	$(x + 1)(x - 2) = 0$ $\therefore x^2 - x - 2 = 0$ The roots are $x = -1$ and 2.	• The polynomial equation $(x + 1)(x - 2) = 0$ has two distinct roots and the graph of the polynomial function $y = (x + 1)(x - 2)$ cuts the x-axis in two distinct points.
ii	$(x + 1)^2 = 0$ or $(x + 1)(x + 1) = 0$ $\therefore x^2 + 2x + 1 = 0$ Roots are $x = -1$ and -1, so the distinct root is $x = -1$. -1 is called a double root of the equation.	• The polynomial equation $(x + 1)^2 = 0$ has a double root at $x = -1$ and the graph of $y = (x + 1)^2$ touches the x-axis at $x = -1$; that is, the x-axis is a tangent to the curve at $x = -1$.
iii	$x^2 + 1 = 0$ has no roots.	• The polynomial equation $x^2 + 1 = 0$ has no (real) roots ($x^2 = -1$ has no solution) and the graph of $y = x^2 + 1$ does not cut the x-axis.

b

i



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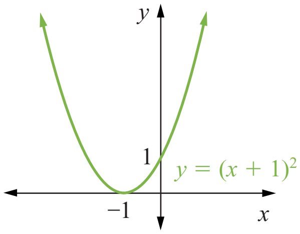

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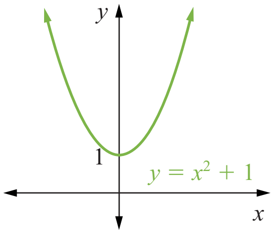
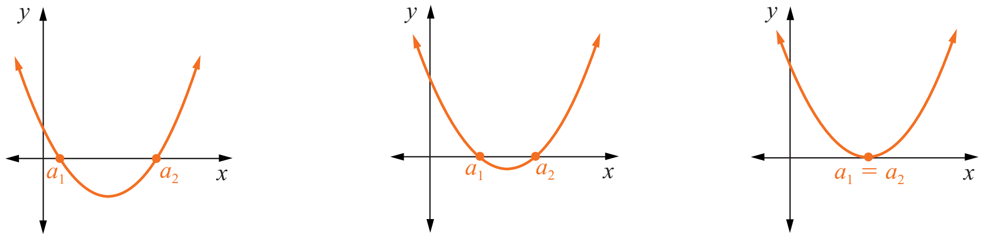


NUMBER & ALGEBRA

Graphically the case of a double root (two equal roots) may be considered as the limiting position as two distinct roots approach each other, as shown.



Exercise 15J

1 **a** Give an example of a polynomial of degree 0.
b Sketch the graph of the corresponding function.
c How many zeros does the polynomial have?

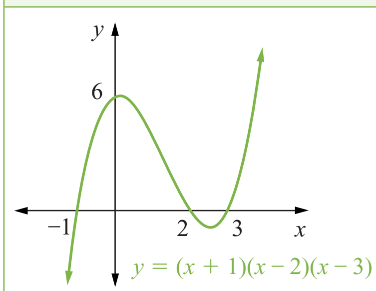
2 **a** Give an example of a polynomial of degree 1.
b Sketch the graph of the corresponding function.
c How many zeros does the polynomial have?
d Can a polynomial of degree 1 have no zeros?

3 **a** What is the maximum number of zeros a polynomial of degree 2 can have?
b Give an example of a polynomial of degree 2 with:
i two distinct zeros **ii** one distinct zero **iii** no zeros
c Sketch the corresponding polynomial function.

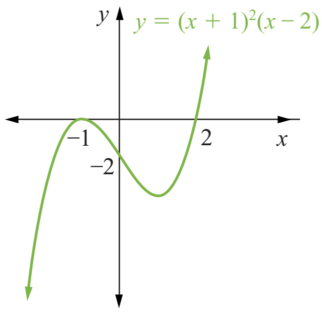
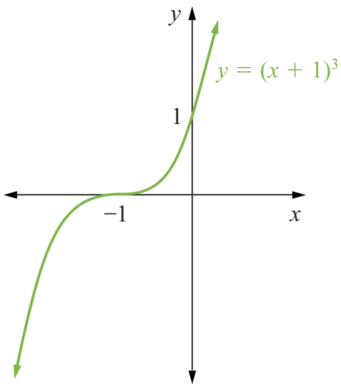
4 **a** What is the maximum number of zeros a polynomial of degree 3 can have?
b Can a polynomial of degree 3 have no zeros?
c Sketch an example of a polynomial of degree 3 with the following number of distinct zeros.
i 3 **ii** 2 **iii** 1
d Give an example of a polynomial equation of degree 3 with the following number of distinct zeros.
i 3 **ii** 2 **iii** 1

EXAMPLE 2

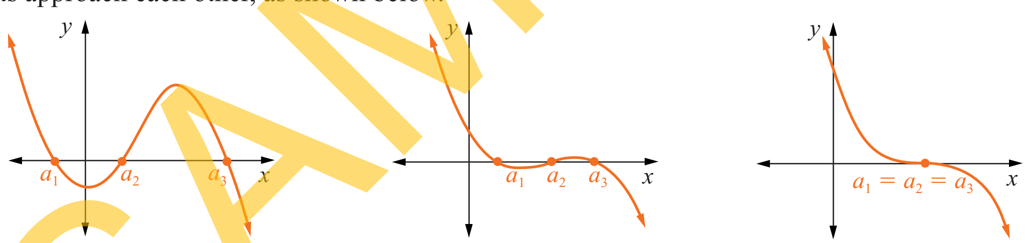
Sketch the following polynomial functions.
a $y = (x + 1)(x - 2)(x - 3)$ **b** $y = (x + 1)^2(x - 2)$ **c** $y = (x + 1)^3$

	Solve	Think/Apply
a		$y = 0$ has three distinct roots: $x = -1$, 2 and 3 These are the x-intercepts. Let $x = 0$: $y = (+1)(-2)(-3) = 6$ This is the y-intercept. The leading term (after expanding) will be x^3. $\therefore$ As $x \to +\infty, y \to +\infty$ and as $x \to -\infty, y \to -\infty$

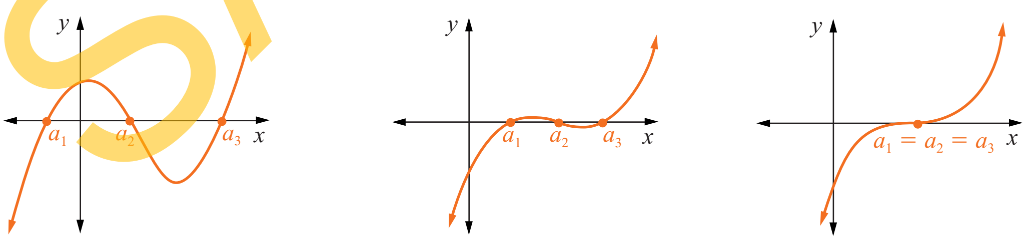
EXAMPLE 2 CONTINUED

	Solve	Think/Apply
b		$y = 0$ has three roots $x = -1, -1$ and 2 (from $(x + 1)(x + 1)(x - 2) = 0$), but these are not all distinct. Distinct roots are $x = -1$ and 2. These are the x-intercepts. As -1 is a double root of the equation, the graph will touch the x-axis at $x = -1$; that is, the x-axis is a tangent to the curve at $x = -1$. Let $x = 0$: $y = -2$ The leading term is x^3. $\therefore$ As $x \to +\infty, y \to +\infty$ and as $x \to -\infty, y \to -\infty$
c		$y = 0$ has three roots $x = -1, -1$ and -1 (from $(x + 1)(x + 1)(x + 1) = 0$), but these are obviously all equal. The distinct root is $x = -1$. This is the only x-intercept. -1 is called a triple root of the equation. Let $x = 0$: $y = 1$ The leading term is x^3. $\therefore$ As $x \to +\infty, y \to +\infty$ and as $x \to -\infty, y \to -\infty$ *Note:* When $x = -1.1$, $y < 0$ and when $x = -0.9$, $y > 0$. So y changes sign as the curve passes through $x = -1$; that is, the curve cuts the x-axis at $x = -1$. Remember this is the curve $y = x^3$ translated 1 unit to the left.

Graphically, the case of a triple root (three equal roots) may be considered as the limiting position as three distinct roots approach each other, as shown below.

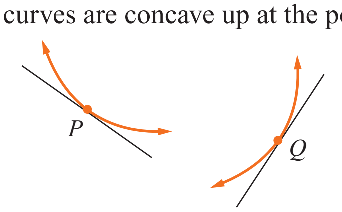


or

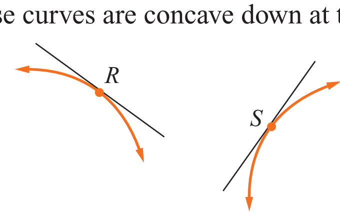


Note that the concavity of the curve changes as we pass through the triple root from left to right along the curve. A curve is concave up at a point if it lies above the tangent at that point, and is concave down at a point if it lies below the tangent at that point.

These curves are concave up at the points P and Q.



These curves are concave down at the points R and S.



In Example 2 parts **a**, **b** and **c**, the polynomials are of degree 3 with 3, 2 and 1 distinct zeros, respectively. The question arises as to whether it is possible for a polynomial of degree 3 to have no zeros.

Consider the following argument.

Let $y = ax^3 + bx^2 + cx + d$. For x very large, positive or negative, the graph approaches $y = ax^3$.

If $a > 0$, as $x \to +\infty$, $y \to +\infty$ and as $x \to -\infty$, $y \to -\infty$.

As y is a continuous function, the curve must cut the x-axis at least once.

If $a < 0$, as $x \to +\infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to +\infty$.

Again, as y is a continuous function the curve must cut the x-axis at least once; that is, any polynomial function of degree 3 must cut the x-axis at least once. Hence a polynomial of degree 3 must have at least one zero.

The same argument can be used for any polynomial of odd degree.

Summary

1 To sketch $y = P(x)$:
- Find the x-intercepts by finding the zeros of $P(x)$.
- Find the y-intercept by putting $x = 0$. This is the constant term of $P(x)$.

2 The significance of double and triple roots:
- If the equation $P(x) = 0$ has a double root at $x = a$, then the x-axis is a tangent to the curve $y = P(x)$ at $x = a$.
- If $P(x) = 0$ has a triple root at $x = b$, then the curve cuts the x-axis and changes concavity at $x = b$.

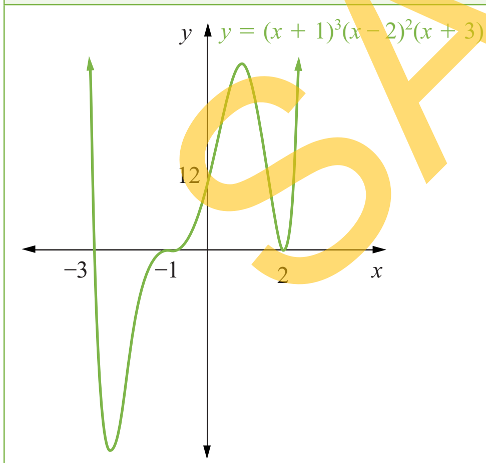
3 If $P(x)$ is of degree n, then at most it can have n zeros.

4 If $P(x)$ is of odd degree, then it has at least one zero.

EXAMPLE 3

Sketch the function $y = (x + 1)^3(x - 2)^2(x + 3)$.

Solve



Think/Apply

Zeros are -3, -1 and 2 with a double root at $x = 2$.
$\therefore$ The x-axis is a tangent to the curve at $x = 2$.
There is a triple root at $x = -1$.
The curve cuts the x-axis at $x = -1$ and changes concavity at this point.
Let $x = 0$: $y = (1)^3(-2)^2(3) = 12$
The y-intercept $= 12$.
The leading term is x^6.
$\therefore$ As $x \to +\infty$, $y \to +\infty$ and as $x \to -\infty$, $y \to +\infty$.

5 Sketch the following polynomials.

a $y = (x + 1)(x - 2)^2$ **b** $y = (x + 2)^3$ **c** $y = (x + 2)(x - 3)^3$
d $y = -x^2(x - 1)^3$ **e** $y = x^3(x - 1)^2$ **f** $y = -x^3(x - 2)^2(x + 3)$

6 Use the factor theorem to find the zeros of the following polynomials and hence sketch each function.

a $P(x) = 2x^3 - 18x^2 + 30x + 50$ **b** $P(x) = x^3 - 3x - 2$

c $P(x) = -x^3 + 2x^2 - x$ **d** $P(x) = x^4 + x^3$

e $P(x) = -x^3 + 6x^2 - 12x + 8$ **f** $P(x) = x^4 - x^3 - 3x^2 + 5x - 2$

7 Write an equation that could represent the following polynomials and sketch the corresponding polynomial functions.

a **i** degree = 1, single root at $x = 3$, constant term = -6

ii degree = 1, single root at $x = 3$, constant term = $+6$

b **i** degree = 2, single roots at $x = -1, 4$, constant term = -4

ii degree = 2, single roots at $x = -1, 4$, constant term = $+4$

c degree = 2, double root at $x = 3$, leading coefficient = -2

d degree = 3, single roots at $x = -1, 0, 1$, monic

e degree = 3, single root at $x = -2$, double root at $x = 1$, monic

f degree = 3, triple root at $x = 4$, leading coefficient = 2

g degree = 4, single root at $x = -3$, triple root at $x = 2$, constant term = -24

h degree = 4, double root at $x = -2$, double root at $x = 2$, monic

8 **a** Produce your own sketch of a polynomial function.

b Describe the main features of your sketch to another student so that they can reproduce your sketch without looking at it.

c Compare your graph with that of your partner. Are they identical? Discuss any differences.

d Swap roles and repeat parts **a** to **c**.

9 **a** Sketch $y = (x + 2)(x - 4)$ showing the x-intercepts, the y-intercept and the coordinates of the turning point.

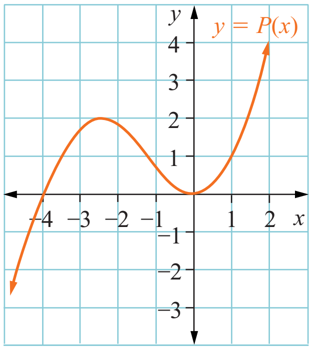
b Use your sketch in part **a** to help you sketch:

i $y = 2(x + 2)(x - 4)$ **ii** $y = \frac{1}{2}(x + 2)(x - 4)$

iii $y = (x + 2)(x - 4) + 1$ **iv** $y = (x + 2)(x - 4) - 2$

v $y = -(x + 2)(x - 4)$ **vi** $y = (-x + 2)(-x - 4)$ (*Note:* $y = P(-x)$)

10 A sketch of $y = P(x)$ is given.



a Use it to sketch the following polynomials.

i $y = 2P(x)$

ii $y = \frac{1}{2}P(x)$

iii $y = P(x) + 2$

iv $y = P(x) - 1$

v $y = -P(x)$

vi $y = P(x - 2)$

vii $y = P(-x)$

b Describe in words the relationship between the graph of $y = P(x)$ and the graph of each of the polynomials in part **a**.

11 Using a graphics calculator, sketch $y = x^3 + x^2 + x + 1$ and $y = x^3 + x^2 + x$. Discuss the similarities and differences in the graphs.

Language in mathematics

1 Describe the effect on the graph of $y = x^n$ of including a constant k as follows.

a $y = kx^n$ **b** $y = x^n + k$ **c** $y = (x - k)^n$

2 Describe line symmetry and point symmetry by reference to graphs of $y = x^n$.

3 Describe the relationship between these graphs.

a $y = x^n$ and $y = -x^n$ **b** $y = x^n$ and $y = (-x)^n$

4 Consider these terms.

a translate **b** function

i State their mathematical meaning.

ii Write their ordinary English meaning.

5 Explain in your own words the meaning of the following terms.

a coefficient **b** concave

c distinct **d** infinity

6 Define the following terms.

a polynomial **b** degree **c** leading term

d leading coefficient **e** constant term **f** monic polynomial

7 Using an example of the long division algorithm, define the terms 'divisor', 'dividend', 'quotient' and 'remainder'.

8 Three of the words below are spelt incorrectly. Give the correct spelling for these words.

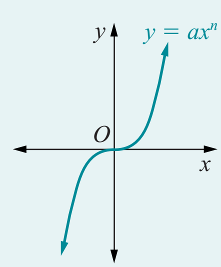
compress, factoor, triple, significance, algebracally, skech

Terms

algebraically	algorithm	compress	concave	constant
coordinates	degree of polynomial	distinct	dividend	divisor
double root	function	graphically	horizontally	infinity
intercept	intersection	leading coefficient	leading term	monic
polynomial	quotient	reflection	relationship	remainder
significant	sketch	stretch	symmetry	transformation
translate	triple root	unique	vertically	zero

Check your skills

1 Given the graph of $y = ax^n$, which of the following statements could not be true?



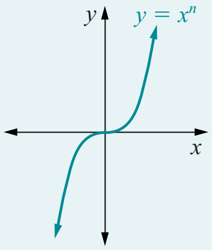
A n is even.

B n is odd.

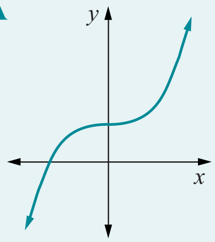
C a is positive.

D The curve is symmetrical about O.

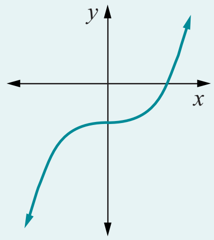
2 This is the graph of $y = x^n$. Which of the following could be the graph of $y = 2x^n - 1$?



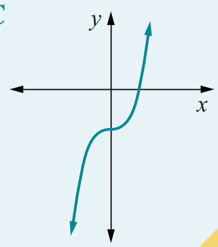
A



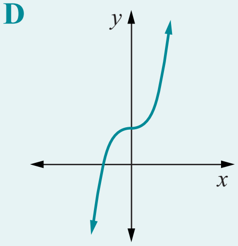
B



C

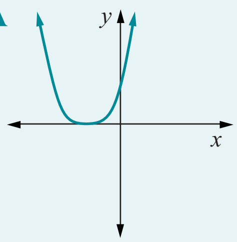


D

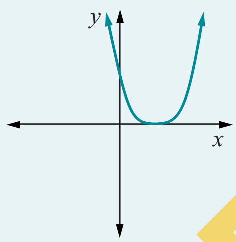


3 Which of the following could be the graph of $y = (x + 2)^4$?

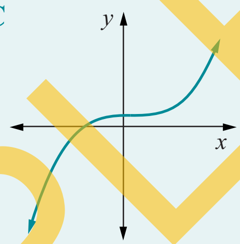
A



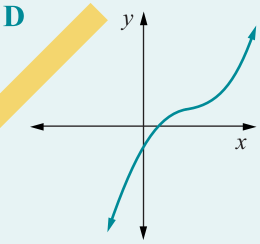
B



C

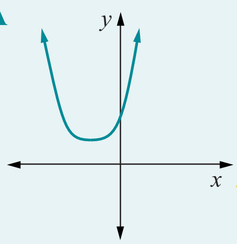


D

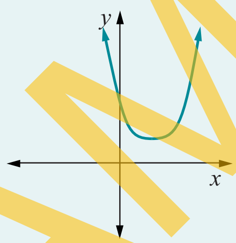


4 Which of the following could be the graph of $y = -(x - 1)^4 + 2$?

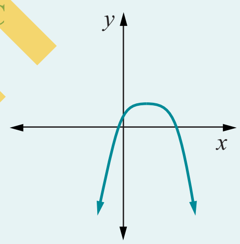
A



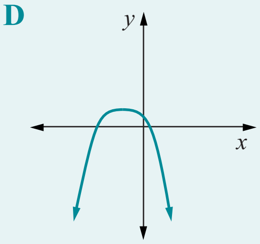
B



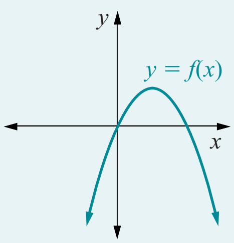
C



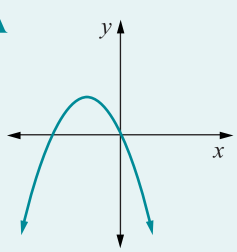
D



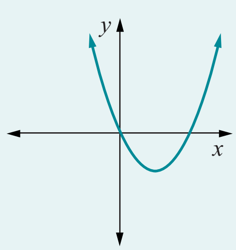
5 Given $y = f(x)$, which of the following could be the graph of $y = -f(x)$?



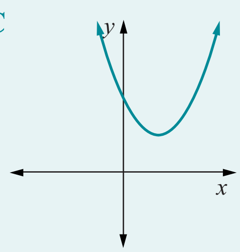
A



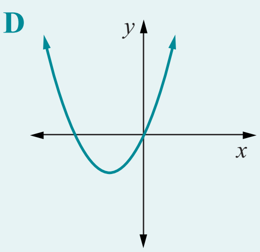
B



C



D



6 Which of the following is not symmetrical about the y-axis?

A $y = 3x^{10}$ **B** $y = -3x^{10}$ **C** $y = 3x^{10} + 1$ **D** $y = 3x^{10} + x^3$

7 Which of the following expressions is not a polynomial?

A $\dfrac{1}{2x^3 + 3x^2 - 4x + 5}$ **B** $x^4 + x^2 + 1$ **C** 15 **D** $x + 3$

8 A polynomial $P(x)$ with degree = 2, constant term = 2 and $P(-1) = 4$ could be:

A $2x^2 + 2x$ **B** $x^2 - x + 2$ **C** $x^2 + 2$ **D** $x^2 + x + 2$

9 Which of the statements below is always true?

i The degree of the $P(x) \pm Q(x)$ equals the degree of $P(x)$ or $Q(x)$, whichever is the greater.

ii The degree of $P(x) \times Q(x)$ equals the degree of $P(x)$ multiplied by the degree of $Q(x)$.

A **i** only **B** **ii** only **C** both **i** and **ii** **D** neither **i** nor **ii**

10 When $x^3 - 4x^2 - 5x + 6$ is divided by $x - 2$ the quotient is:

A $x^2 + 2x - 1$ **B** $x^2 - 6x - 17$ **C** $x^2 + 6x + 7$ **D** $x^2 - 2x - 9$

11 When $P(x) = 2x^3 - 4x^2 + 3x + 2$ is divided by $(x + 3)$ the remainder is:

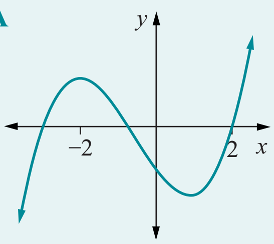
A -97 **B** -25 **C** 11 **D** 83

12 If $(x - 1)$ and $(x + 2)$ are factors of $P(x) = 3x^3 + ax^2 + bx + 4$, then the values of a and b are respectively:

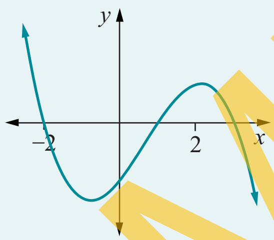
A 3 and -10 **B** 3 and -4 **C** 1 and -8 **D** 1 and 8

13 Which of the following could be the graph of $y = (3 - x)(x + 2)(x - 1)$?

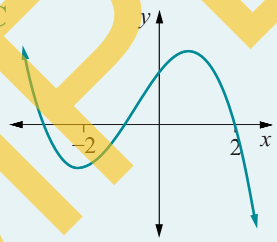
A



B



C



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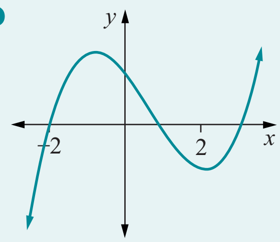


14 Which of the statements below is always true?

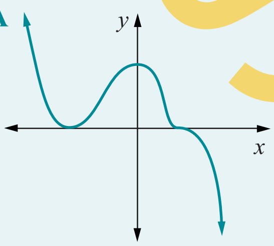
i If $P(x)$ is of degree n, then $P(x)$ has n zeros.

ii If $P(x)$ is of odd degree, then $P(x)$ has at least one zero.

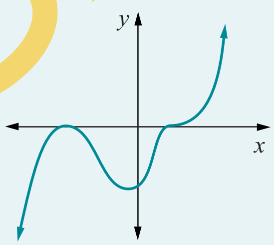
A **i** only **B** **ii** only **C** both **i** and **ii** **D** neither **i** nor **ii**

15 Which of the following could be the graph of $y = (x - 1)^2(x + 2)^3$?

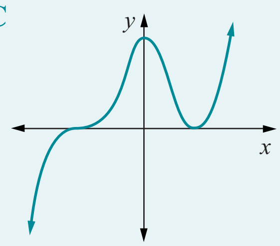
A



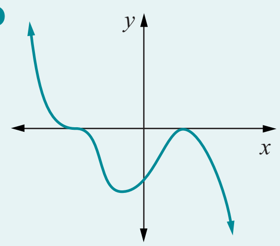
B



C



D



If you have any difficulty with these questions, refer to the examples and questions in the sections listed in the table.

Question	1–5	6	7, 8	9	10	11	12	13	14	15
Section	A	B	C	D	E	F	G	H	I	J

15A Review set

1 On the same diagram, draw neat sketches of:

a **i** $y = x^2$ **ii** $y = 3x^2$ **iii** $y = \frac{1}{2}x^2$ **iv** $y = -x^2$

b **i** $y = x^2$ **ii** $y = x^2 + 3$ **iii** $y = x^2 - 3$

c **i** $y = x^2$ **ii** $y = (x - 2)^2$ **iii** $y = (x + 1)^2$

d **i** $y = x^2$ **ii** $y = 2x^2 - 1$

e **i** $y = x^2$ **ii** $y = (x - 1)^2 - 3$

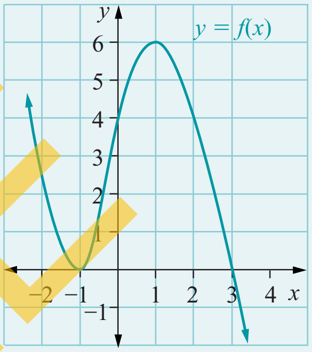
f **i** $y = x^2$ **ii** $y = -(x + 1)^2 + 2$ **iii** $y = -\frac{1}{2}(x + 1)^2 + 2$

2 Use the graph of $y = f(x)$ given to draw neat sketches of:

a $y = 2f(x)$ **b** $y = f(x) + 2$

c $y = f(x - 2)$ **d** $y = f(x + 2)$

e $y = -f(x)$ **f** $y = f(-x)$



3 Determine whether the following functions are symmetrical about the y-axis.

a $f(x) = -3x^2$ **b** $f(x) = 2x^3$ **c** $f(x) = x^2 - 2x + 3$

4 State whether or not the following are polynomials.

a $4x^3 - 3x^2 + 7x + \frac{2}{3}$ **b** $x^2 + 3x + \frac{2}{x}$ **c** 3 **d** $3^x + 2^x + 5$

5 For each of the following polynomials state:

i the degree **ii** the leading term **iii** the leading coefficient

iv the constant term **v** whether or not the polynomial is monic.

a $2x^3 - 5x^2 - 7x - 2$ **b** $4x + 2x^3 - 2x^5$

6 Given $P(x) = 2x^3 - x^2 + 7$ and $Q(x) = x^3 + 3x^2 - 4x - 2$, find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$

7 The degree of $P(x) + Q(x)$ equals the degree of $P(x)$ or $Q(x)$, whichever is the greater. This statement is true:

A never **B** sometimes **C** always

8 Given $P(x) = x - 1$ and $Q(x) = 4x^3 + 2x^2 - 3x + 5$, find:

a $P(x) \times Q(x)$

b $Q(x) \div P(x)$ Express the result in the form $Q(x) = P(x) \times A(x) + R$.

9 **a** Find the remainder when $x^3 - 2x^2 - 5x + 1$ is divided by $x - 2$.

b When $P(x) = kx^2 - 5x + 3$ is divided by $(x + 1)$, the remainder is 20. Find k.

10 **a** Show that $x - 2$ is a factor of $P(x) = x^3 + 3x^2 - 4x - 12$.

b Hence find all the linear factors of $P(x)$.

c State the zeros of $y = P(x)$.

d Draw a neat sketch of $y = P(x)$.

11 Sketch the following polynomials.

a $y = (x - 1)^2(x + 2)^3$ **b** $y = -(x + 3)^3(x - 2)^2$

15B Review set

1 On the same diagram, draw neat sketches of:

a **i** $y = x^3$ **ii** $y = 2x^3$ **iii** $y = \frac{1}{2}x^3$ **iv** $y = -x^3$

b **i** $y = x^3$ **ii** $y = x^3 + 1$ **iii** $y = x^3 - 2$

c **i** $y = x^3$ **ii** $y = (x - 1)^3$ **iii** $y = (x + 2)^3$

d **i** $y = x^3$ **ii** $y = 3x^3 + 1$

e **i** $y = x^3$ **ii** $y = (x - 2)^3 - 1$

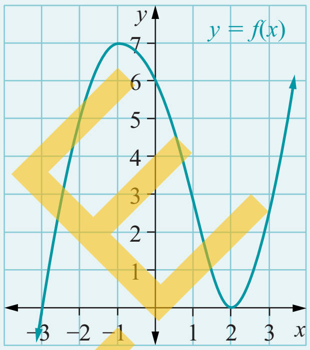
f **i** $y = x^3$ **ii** $y = -(x - 1)^3 + 2$ **iii** $y = -\frac{1}{2}(x - 1)^3 + 2$

2 Use the graph of $y = f(x)$ given to draw neat sketches of:

a $y = 2f(x)$ **b** $y = f(x) + 2$

c $y = f(x - 2)$ **d** $y = f(x + 2)$

e $y = -f(x)$ **f** $y = f(-x)$



3 Determine whether the following functions are symmetrical about the y-axis.

a $f(x) = -4x^3$ **b** $f(x) = x^4 + x^2$ **c** $f(x) = x^4 + x^2 - 1$

4 State whether or not the following are polynomials.

a $1 + 4x + x^2 + x^3$ **b** $x^3 + \sqrt{x^3}$ **c** 42 **d** $\dfrac{2}{x^2 + 8x + 11}$

5 For each of the following polynomials state:

i the degree **ii** the leading term **iii** the leading coefficient

iv the constant term **v** whether or not the polynomial is monic.

a $3 + 2x - x^2 + 5x^3$ **b** $x^4 + 2x^3 + 3x^2 + 4x + 5$

6 Given $P(x) = 4x^3 + 7x^2 - 2x + 11$ and $Q(x) = -2x^3 + x^2 + 7x - 4$, find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$

7 How is the degree of $P(x) \pm Q(x)$ related to the degree of $P(x)$ and $Q(x)$?

8 Given $P(x) = x + 2$ and $Q(x) = x^3 + 6x^2 - 4x + 5$, find:

a $P(x) \times Q(x)$

b $Q(x) \div P(x)$ Express the result in the form $Q(x) = P(x) \times A(x) + R$.

9 **a** Find the remainder when $2x^3 + 4x^2 - 11x + 3$ is divided by $x + 1$.

b When $P(x) = x^3 + kx^2 - 2x - 1$ is divided by $x - 2$, the remainder is 6. Find k.

10 **a** Show that $(x + 1)$ is a factor of $P(x) = x^3 + 2x^2 - 11x - 12$.

b Hence find all the linear factors of $P(x)$.

c State the zeros of $y = P(x)$.

d Draw a neat sketch of $y = P(x)$.

11 Sketch the following polynomials.

a $y = (x + 2)^3(x - 3)^2$ **b** $y = -(x - 1)^3(x + 2)^2$

15C Review set

1 On the same diagram, draw neat sketches of:

a **i** $y = x^4$ **ii** $y = 2x^4$ **iii** $y = \frac{1}{3}x^4$ **iv** $y = -x^4$

b **i** $y = x^4$ **ii** $y = x^4 + 1$ **iii** $y = x^4 - 2$

c **i** $y = x^4$ **ii** $y = (x - 3)^4$ **iii** $y = (x + 3)^4$

d **i** $y = x^4$ **ii** $y = 2x^4 - 3$

e **i** $y = x^4$ **ii** $y = (x - 1)^4 + 2$

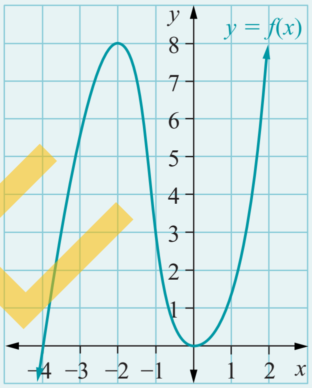
f **i** $y = x^4$ **ii** $y = -(x + 1)^4 + 1$ **iii** $y = -3(x + 1)^4 - 1$

2 Use the graph of $y = f(x)$ given to draw neat sketches of:

a $y = 2f(x)$ **b** $y = f(x) + 2$

c $y = f(x - 2)$ **d** $y = f(x + 2)$

e $y = -f(x)$ **f** $y = f(-x)$



3 Determine whether the following functions are symmetrical about the y-axis.

a $f(x) = 2x^{10}$ **b** $f(x) = x^3 - x + 1$ **c** $f(x) = x^4 - x^2$

4 State whether or not the following are polynomials.

a $x + \sqrt{x} + 1$ **b** $\frac{1}{2}x^3 - \frac{2}{3}x^2 + \frac{4}{5}x - \frac{1}{4}$ **c** $3x + \frac{3}{x}$ **d** 14

5 For each of the following polynomials state:

i the degree **ii** the leading term **iii** the leading coefficient

iv the constant term **v** whether or not the polynomial is monic.

a $1 - x + x^2 - 2x^3$ **b** $x^4 - 6x^2 + 3$

6 Given $P(x) = 4x^3 + x^2 - 5x + 7$ and $Q(x) = 3x^3 - 7x^2 - 11x - 12$ find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$

7 The degree of $P(x) + Q(x)$ equals the degree of $P(x)$ plus the degree of $Q(x)$. This statement is true:

A never **B** sometimes **C** always

8 Given $P(x) = x - 2$ and $Q(x) = 3x^3 - 4x^2 + 9x - 3$, find:

a $P(x) \times Q(x)$

b $Q(x) \div P(x)$ Express the result in the form $Q(x) = P(x) \times A(x) + R$.

9 **a** Find the remainder when $2x^3 + 11x^2 - 8x + 3$ is divided by $x + 2$.

b When $P(x) = x^3 - kx^2 + 4x - 3$ is divided by $x - 1$, the remainder is 10. Find k.

10 **a** Show that $(x + 3)$ is a factor of $P(x) = x^3 - 13x - 12$.

b Hence find all the linear factors of $P(x)$.

c State the zeros of $y = P(x)$.

d Draw a neat sketch of $y = P(x)$.

11 Sketch the following polynomials.

a $y = x^2(x - 2)^3$ **b** $y = -x^3(x + 1)^2$

15D Review set

1 On the same diagram, draw neat sketches of:

a **i** $y = x^5$ **ii** $y = 2x^5$ **iii** $y = \frac{1}{3}x^5$ **iv** $y = -x^5$

b **i** $y = x^5$ **ii** $y = x^5 + 3$ **iii** $y = x^5 - 3$

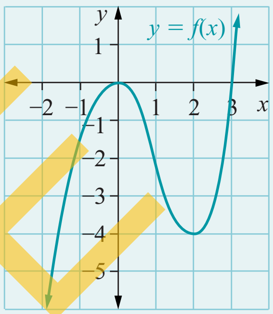
c **i** $y = x^5$ **ii** $y = (x - 2)^5$ **iii** $y = (x + 2)^5$

d **i** $y = x^5$ **ii** $y = 3x^5 - 1$

e **i** $y = x^5$ **ii** $y = (x - 1)^5 + 2$

f **i** $y = x^5$ **ii** $y = -(x + 1)^5 - 3$ **iii** $y = -\frac{1}{2}(x + 1)^5 - 3$

2 Use the graph of $y = f(x)$ given to draw neat sketches of:



a $y = 2f(x)$ **b** $y = f(x) + 2$

c $y = f(x - 2)$ **d** $y = f(x + 2)$

e $y = -f(x)$ **f** $y = f(-x)$

3 Determine whether the following functions are symmetrical about the y-axis.

a $f(x) = 4x^2$ **b** $y = -3x^4$ **c** $y = x^4 - 3x^2$

4 State whether or not the following are polynomials.

a $\frac{5}{8}x^2 - \frac{3}{5}x + \frac{2}{3}$ **b** $x + \frac{1}{\sqrt{x}}$ **c** $\frac{1}{x^3 + x^2 - x - 1}$ **d** 21

5 For each of the following polynomials state:

i the degree **ii** the leading term **iii** the leading coefficient

iv the constant term **v** whether or not the polynomial is monic.

a $x^4 + 3x^2 - 1$ **b** $1 - 2x + 3x^2 - 4x^3$

6 Given $P(x) = x^4 + 3x^3 - 2x^2 + 3x - 11$ and $Q(x) = -2x^3 + x^2 - 4x + 10$ find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$

7 The degree of $P(x) \times Q(x)$ equals the degree of $P(x)$ multiplied by the degree of $Q(x)$. This statement is true:

A never **B** sometimes **C** always

8 Given $P(x) = x + 3$ and $Q(x) = x^3 - 12x^2 + 5x - 3$, find:

a $P(x) \times Q(x)$

b $Q(x) \div P(x)$ Express the result in the form $Q(x) = P(x) \times A(x) + R$.

9 **a** Find the remainder when $P(x) = 2x^3 - x^2 + x - 4$ is divided by $x + 3$.

b When $P(x) = kx^3 - 2x^2 + 2x - 1$ is divided by $(x + 2)$, the remainder is 1. Find k.

10 **a** Show that $x + 3$ is a factor of $P(x) = x^3 - 2x^2 - 9x + 18$.

b Hence find all the linear factors of $P(x)$.

c State the zeros of $y = P(x)$.

d Draw a neat sketch of $y = P(x)$.

11 Sketch the following polynomials.

a $y = (x + 2)^2(x - 1)^2$ **b** $y = (x + 1)^3(x - 3)^2$