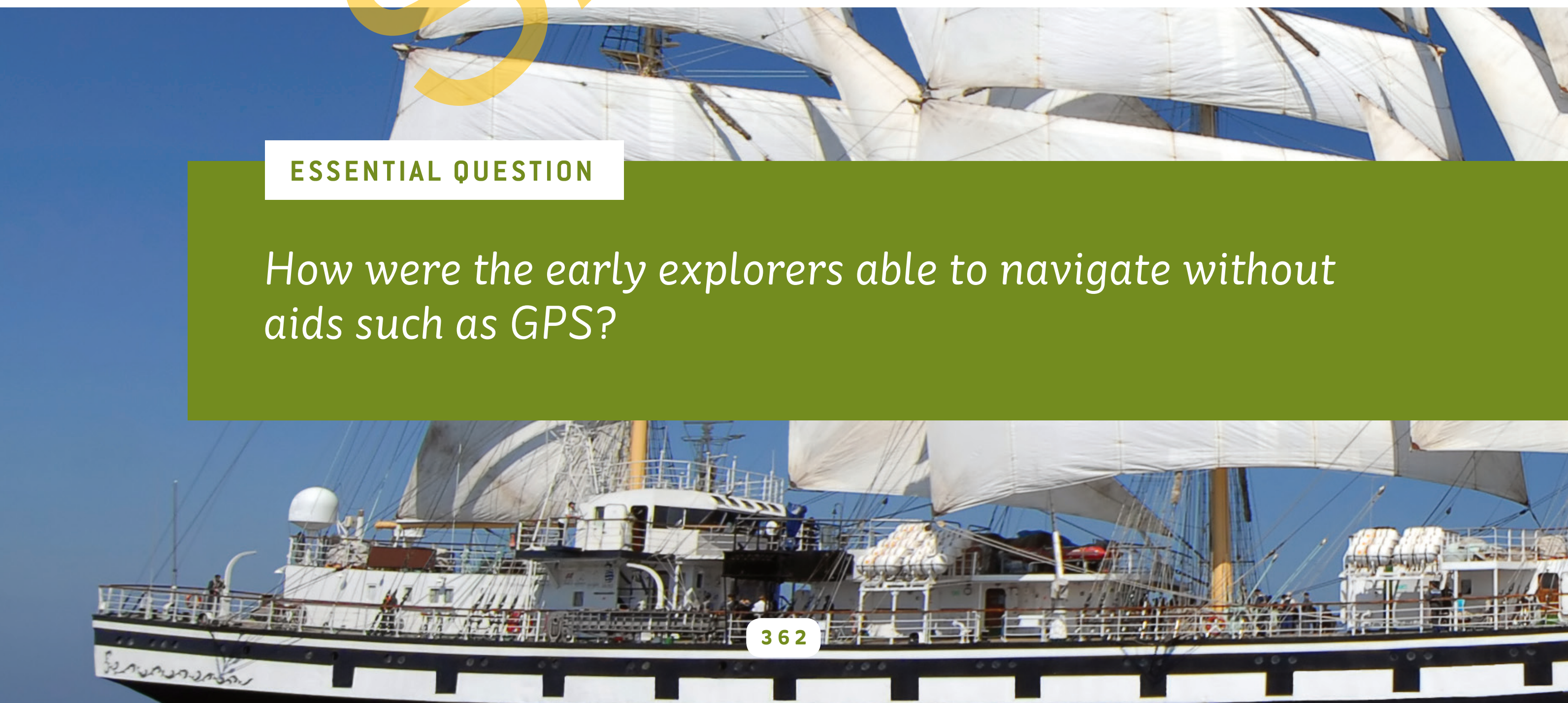
8 PYTHAGORAS' THEOREM AND TRIGONOMETRY



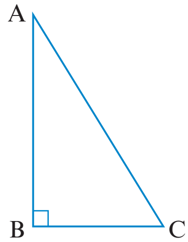
ESSENTIAL QUESTION

How were the early explorers able to navigate without aids such as GPS?

8A **1** Calculate the following, correct to one decimal place.

- **a** $\sqrt{40}$ **b** $\sqrt{14.9}$
- **c** $\sqrt{22.04}$ **d** $\sqrt{145.3}$

Questions **2–6** relate to this figure.



8A **2** Which line segment is the hypotenuse?

8A **3** If AB = 9 mm and AC = 10 mm, give the length of BC, correct to the nearest millimetre.

8B **4** How can sin ∠BAC be written?

A $\frac{BC}{AB}$ **B** $\frac{BC}{AC}$ **C** $\frac{AB}{AC}$ **D** $\frac{AC}{BC}$

8B **5** What is the ratio $\frac{AB}{BC}$?

- **A** tan ∠BAC **B** sin ∠BAC
- **C** cos ∠ACB **D** tan ∠ACB

8C **6** Write a ratio for:

- **a** cos ∠BAC
- **b** cos ∠ACB

8C **7** What is the value of tan 68°, correct to two decimal places?

A 2.48 **B** 0.93 **C** 0.37 **D** 0.40

8C **8** What is the value of $\cos^{-1} 0.45$, correct to the nearest degree?

A 27° **B** 24° **C** 63° **D** 66°

8D **9** Calculate the value of each expression, correct to two decimal places.

- **a** $3.2 \times \cos 43°$ **b** $\frac{67}{\tan 78°}$
- **c** $\frac{4.5}{\cos 62°}$ **d** $0.7 \times \sin 85°$

8D **10** Give definitions for:

- **a** angle of elevation
- **b** angle of depression
- **c** compass direction
- **d** true bearing.

8D **11** **a** Write the true bearing that is equivalent to each compass bearing.

- **i** NE **ii** SSE
- **iii** ENE **iv** WNW

b Write the true bearing for each compass bearing.

- **i** S30°E **ii** S28°W
- **iii** S2°E **iv** N8°W

8D **12** If one of the acute angles in a right-angled triangle is 27°, what is the value of the other acute angle?

8E 10A **13** How many faces are on the surface of each of these solids?

- **a** rectangular prism
- **b** triangular-based pyramid

8E 10A **14** How many edges are there on each of the solids in question **13**?

8F 10A **15** **a** Draw a scalene triangle.

b Describe how you would calculate the area of the triangle.

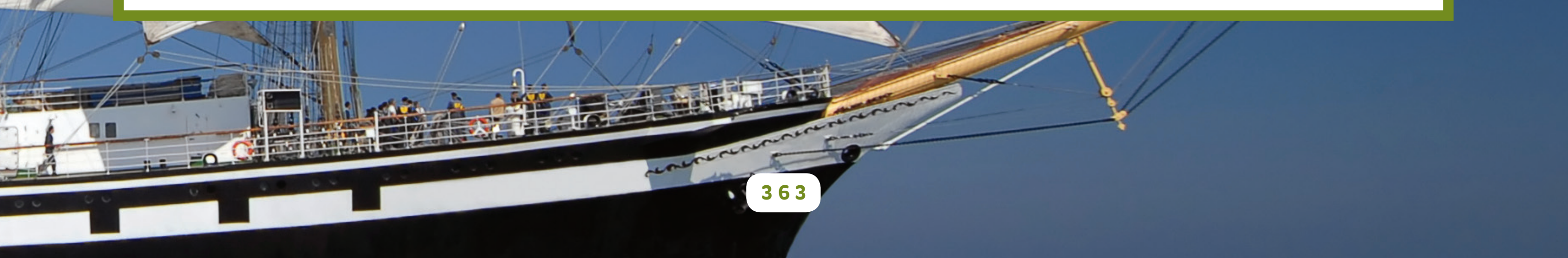
c Label the height and base on your figure, and give a formula for its area.

8H 10A **16** Draw a diagram of the Cartesian plane.

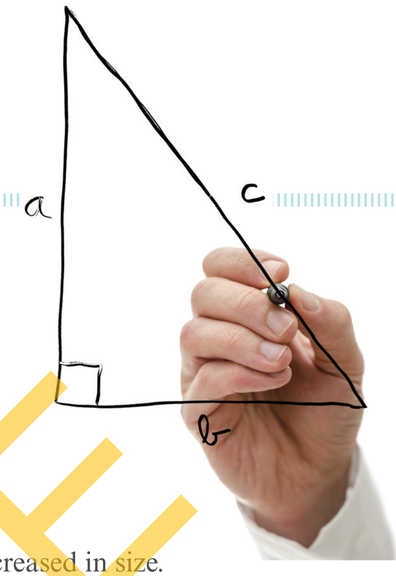
- **a** Label the quadrants.
- **b** Indicate the signs of x and y (positive or negative) in each of the quadrants.

8H 10A **17** Draw a circle. Label:

- **a** its centre **b** a radius
- **c** an arc **d** the circumference.



8A Finding lengths using Pythagoras' Theorem



Start thinking!

Pythagoras' Theorem gives the relationship between the lengths of the sides in a right-angled triangle.

1 **a** Use the figure to state this relationship in words.
  b What is the name given to side c?
  c Explain why side c is always the longest side in a right-angled triangle.

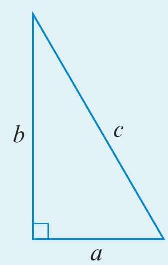
2 Imagine what happens to this triangle if the right angle is increased or decreased in size.
  a First, imagine the right angle is increased to greater than 90°.
    i Draw a sketch of this figure. **ii** Explain what happens to the length of side c.
    iii Adjust Pythagoras' Theorem to write an algebraic relationship between the lengths of the sides.
  b Next, imagine the right angle is decreased to less than 90°.
    i Draw a sketch of this figure. **ii** Explain what happens to the length of side c.
    iii Adjust Pythagoras' Theorem to write an algebraic relationship between the lengths of the sides.

3 Explain how you could use Pythagoras' Theorem to determine whether a triangle is, in fact, right-angled.

4 The ratio 3:4:5 is sometimes mentioned when working with Pythagoras' Theorem. Explain what this ratio refers to.

KEY IDEAS

- When two of the three lengths of a right-angled triangle are known, Pythagoras' Theorem can be used to find the third length.
- Pythagoras' Theorem states that, in any right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.

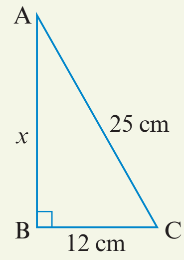


- The Theorem is stated algebraically as $c^2 = a^2 + b^2$.
- If c represents the length of the longest side of a triangle, the triangle is:
  - right-angled if $c^2 = a^2 + b^2$
  - obtuse-angled if $c^2 > a^2 + b^2$
  - acute-angled if $c^2 < a^2 + b^2$.
- **Pythagorean triads** (also called **triples**) are sets of three whole numbers which satisfy Pythagoras' Theorem, such as 3, 4 and 5.

EXERCISE 8A Finding lengths using Pythagoras' Theorem

EXAMPLE 8A-1 Using Pythagoras' Theorem to calculate an unknown side length (decimal value)

Calculate the unknown side length in this triangle, correct to one decimal place.



THINK	WRITE
1 Since the triangle is right-angled, state Pythagoras' Theorem and define a, b and c.	$c^2 = a^2 + b^2$ $a = 12$ cm, $b = x$, $c = 25$ cm
2 Substitute the known values into the equation and simplify.	$25^2 = 12^2 + x^2$ $625 = 144 + x^2$
3 Solve for x by first subtracting 144 from both sides of the equation and then finding the square root.	$x^2 = 625 - 144 = 481$ $x = \sqrt{481} \approx 21.9$
4 Answer the question, providing the correct units.	The length of the third side is 21.9 cm.

UNDERSTANDING AND FLUENCY

1 Calculate the length of the unknown side in each triangle, correct to one decimal place.

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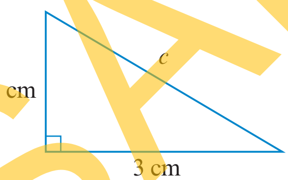

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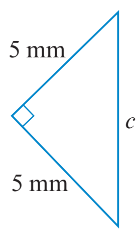

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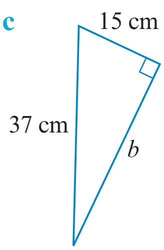

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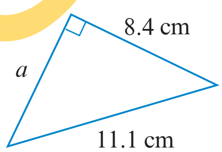

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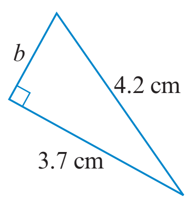

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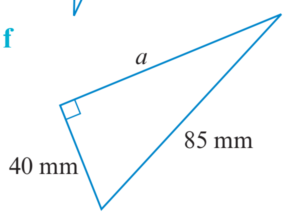

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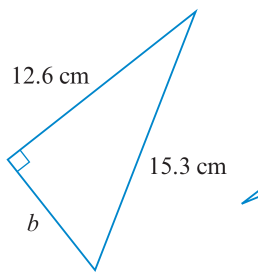

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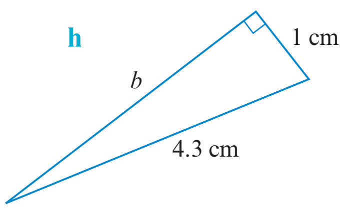
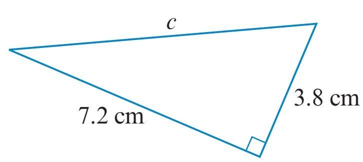
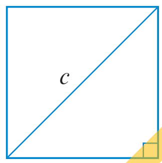


i


EXAMPLE 8A-2 Using Pythagoras' Theorem to calculate an unknown side length (exact value)

A square piece of paper has a side length of 10 cm. What is the length of its diagonal? Give your answer as an exact value, in simplest form.

THINK	WRITE
1 Draw a labelled diagram.	
2 The diagonal is the hypotenuse of the right-angled triangle. Identify a, b and c in the triangle.	$a = 10$, $b = 10$, c is unknown
3 Use Pythagoras' Theorem, leaving the answer in square root form if it is **irrational**.	$c^2 = a^2 + b^2$ $= 10^2 + 10^2$ $= 100 + 100$ $= 200$ $c = \sqrt{200}$
4 Simplify the irrational number. Look for the highest perfect square that is a factor.	$c = \sqrt{100 \times 2}$ $= \sqrt{100} \times \sqrt{2}$ $= 10 \times \sqrt{2}$ $= 10\sqrt{2}$
5 Answer the question, providing the correct units.	The exact length of the diagonal is $10\sqrt{2}$ cm.

UNDERSTANDING AND FLUENCY

2 Calculate the length of the diagonal of a square with each given side length. Give your answer as an exact value in simplest form.

a 5 cm **b** 20 mm **c** 8 cm **d** 25 cm

3 Calculate the length of the diagonal of each square in question **2**, correct to one decimal place.

4 Determine whether each triangle with the given side lengths is right-angled, acute-angled or obtuse-angled.

a 26 cm, 24 cm, 10 cm
b 100 cm, 96 cm, 28 cm
c 3.5 cm, 8.5 cm, 9.3 cm
d 1 cm, 2.4 cm, 2.6 cm
e 7.5 cm, 4.2 cm, 8.5 cm
f 5 cm, 7 cm, 6 cm

5 An equilateral triangle has side lengths of 8 cm.

a Draw a sketch of the triangle. Draw a line from the top vertex that is perpendicular to the base. Label the lengths of all sides of the figure.

b Use Pythagoras' Theorem to calculate the height of the triangle. Give your answer as:

i an exact value

ii an approximate value, correct to one decimal place.

6 A squaring tool is used when making picture frames to ensure the corners of the frames are true right angles.

Imagine you are constructing a square picture frame with side lengths of 54 cm.

a One of the diagonals of the frame measures 78 cm. Explain how you can be sure that the frame is not truly square. Refer to acute and obtuse angles.

b The other diagonal measures 76 cm. Draw a labelled sketch of the frame, showing its true shape.

7 A sheet of writing paper is 21 cm wide and 30 cm long. The envelope used for the paper has a width of 22 cm and a diagonal of 24.6 cm.

a What is the height of the envelope?

b The paper is folded into equal-sized strips along its longer side. Show how to find the *minimum* number of folds necessary to ensure the paper will fit within the envelope.

c If the paper was folded along its shorter side, what would be the minimum number of folds needed?

UNDERSTANDING AND FLUENCY

8 A rectangular running field measures 50 m by 100 m. How much shorter is it to run directly from one corner to the diagonally opposite corner than it is to travel around the outside of the field?

9 Lisa is flying her kite high in the sky. She releases the string to its full length of 55 m, and holds the end of the string at head height. Lisa is 178 cm tall. Assume the string forms a straight line.

Lisa sees the kite flying directly over a tree which she knows is 35 m from where she is standing.

a Draw a diagram to display the scene. Remember to account for Lisa's height.

b Calculate the height the kite is flying above the ground.

10 You know that a triangle with side lengths 3 cm, 4 cm and 5 cm is right-angled. The numbers 3:4:5 are one example of a Pythagorean triad (or triple). There are many more.

a To investigate other triads, start by finding out if multiples of these numbers also satisfy Pythagoras' Theorem. Do each of these sets of numbers satisfy Pythagoras' Theorem?

i 30:40:50 **ii** 6:8:10 **iii** 12:16:20

b Use the triad to check side lengths that are not whole numbers. Show that triangles with the following side lengths are indeed right-angled triangles, using the 3:4:5 triad.

i 1.5 cm, 2 cm, 2.5 cm **ii** 0.6 cm, 0.8 cm, 1 cm **iii** 1.2 cm, 1.6 cm, 2 cm

c What do you conclude from your investigations in parts **a** and **b**?

d All the triangles in parts **a** and **b** are similar triangles. Explain why this is so.

PROBLEM SOLVING AND REASONING

11 There are different formulas which can be used to generate sets of Pythagorean triads. Consider this one. A Pythagorean triad can be written as:

$x^2 - y^2$, $2xy$ and $x^2 + y^2$, where x and y are integers, and $x > y$.

a Which expression will represent the hypotenuse? Will this always be the case?

b Choose values for x and y. Note the restriction placed on these values.

c Calculate values for:

i $x^2 - y^2$ **ii** $2xy$ **iii** $x^2 + y^2$

d Verify that the set of numbers you generate does, in fact, satisfy Pythagoras' Theorem.

e Explain the restriction $x > y$.

f Repeat the procedure to generate another set of Pythagorean triads.

g What values of x and y will generate the triad 3:4:5?

PROBLEM SOLVING AND REASONING

12 A right-angled triangle has two shorter sides and one is twice as long as the other. The hypotenuse is 25 cm long. What are the lengths of the two shorter sides?

13 Josh has an extension ladder which is 9.8 m long. He needs to clean the windows on the outside of a building. The lower part of these windows is 9.8 m from the base of the building, and they are 1.2 m tall. Unfortunately, the closest Josh can place his ladder to the base of the building is 3 m, as there are bushes preventing him from putting it any closer. Josh is 1.7 m tall.

- **a** Draw a diagram to show the situation.
- **b** Calculate the height the ladder reaches up the building.
- **c** Explain whether Josh will be able to clean the windows.

14 The flagpole which sits on Parliament House in Canberra stands 81 m high. The Australian flag flying from the top section of the pole is said to be the largest Australian flag flying in Australia: it's about the size of the side of a double-decker bus! It measures 12.8 m wide, with a diagonal of 14.3 m.

- **a** What is the height of the flag?
- **b** Calculate the perimeter of the flag.
- **c** What is the distance from the base of the flagpole to the lowest point where the flag is attached to the pole?
- **d** Assume the flag is flying flat and directly at right angles to the pole. Calculate the distance from the base of the flagpole to the:
  - **i** lower
  - **ii** higher

  unattached corner of the flag.

15 A cubby house is built in a tree on two levels, the lower level being 10 m above the ground, while the upper level is another 5 m vertically higher. The ladder from the ground to the lower level is 12 m long. Sam wants to construct another ladder to go from the ground 1 m further from the base of the tree to the top level. How long should he make this ladder?

CHALLENGE

16 A ladder rests against a vertical wall, with the top of the ladder 7 m above the base of the wall. When the bottom of the ladder is moved 1 m further away from the base of the wall, the top of the ladder rests against the base of the wall.

- **a** Draw a diagram to illustrate this situation.
- **b** Calculate the length of the ladder.

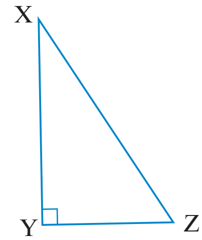
Reflect

Why are utility access holes and covers in roads round in shape rather than square or rectangular?

8B Finding lengths using trigonometry

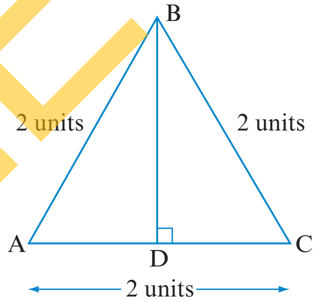
Start thinking!

Trigonometry is the study of triangles and the relationship between their sides and angles.



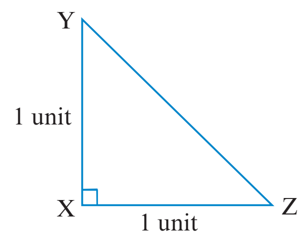
1 Use this right-angled triangle to write ratios for:

a sin ∠YXZ **b** cos ∠YXZ **c** tan ∠YXZ
d sin ∠XZY **e** cos ∠XZY **f** tan ∠XZY



2 Consider the equilateral triangle ABC, which has a perpendicular drawn from the vertex B to the base AC.

a Write the size of: **i** ∠BAD **ii** ∠ABD.
b What is the length of AD?
c Use Pythagoras' Theorem to calculate the length of BD. Leave your answer as an exact number (irrational number in surd form).
d Use your answers from parts **b** and **c** to write values for:
i sin ∠BAD **ii** cos ∠BAD **iii** tan ∠BAD
iv sin ∠ABD **v** cos ∠ABD **vi** tan ∠ABD



3 Consider the right-angled isosceles triangle XYZ. The two equal sides, XY and XZ, are 1 unit in length.

a Use Pythagoras' Theorem to calculate the length of YZ. Leave your answer as an irrational number.
b Write the size of: **i** ∠XYZ **ii** ∠XZY
c Write values (in exact form) for:
i sin ∠XYZ **ii** cos ∠XYZ **iii** tan ∠XYZ
iv sin ∠XZY **v** cos ∠XZY **vi** tan ∠XZY

4 Use your answers from questions **2** and **3** to copy and complete this table.

	30°	45°	60°
sin			
cos			
tan			

KEY IDEAS

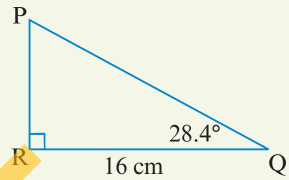
- Trigonometry involves relationships between angles and side lengths in a triangle.
  $\textbf{sin}\ \theta = \dfrac{\textbf{opposite side}}{\textbf{hypotenuse}}$, $\textbf{cos}\ \theta = \dfrac{\textbf{adjacent side}}{\text{hypotenuse}}$, $\textbf{tan}\ \theta = \dfrac{\text{opposite side}}{\text{adjacent side}}$
- The exact values for the trigonometric ratios of 30°, 45° and 60° are as follows:
- Where possible, use exact values for trigonometric ratios. Otherwise, obtain them by using a calculator.

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

EXERCISE 8B Finding lengths using trigonometry

EXAMPLE 8B-1 Using the calculator value of trigonometric ratios to calculate unknown side lengths

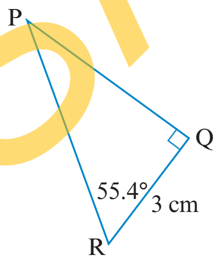
Calculate the lengths of the two unknown sides in triangle PQR.
Give your answers correct to one decimal place.



THINK	WRITE
1 Use the tangent ratio to calculate the length of PR. Simplify to calculate its value.	$\tan 28.4° = \dfrac{PR}{16}$ $\dfrac{PR}{16} = \tan 28.4°$ $16 \times \dfrac{PR}{16} = 16 \times \tan 28.4°$ $PR \approx 8.7$ cm
2 Use the cosine ratio to calculate the length of PQ. Simplify to calculate its value.	$\cos 28.4° = \dfrac{16}{PQ}$ $PQ \times \cos 28.4° = PQ \times \dfrac{16}{PQ}$ $PQ \times \cos 28.4° = 16$ $PQ = \dfrac{16}{\cos 28.4°}$ ≈ 18.2 cm

UNDERSTANDING AND FLUENCY

1 Find the length of the unknown sides in each triangle, correct to one decimal place.

a


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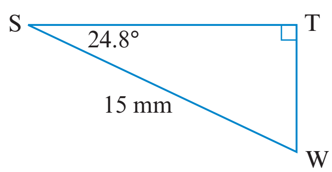

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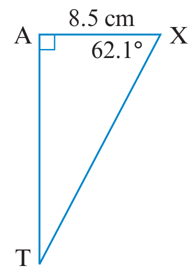

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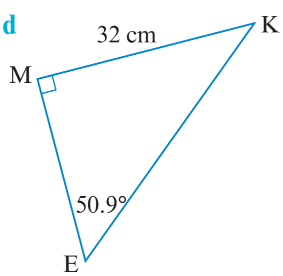

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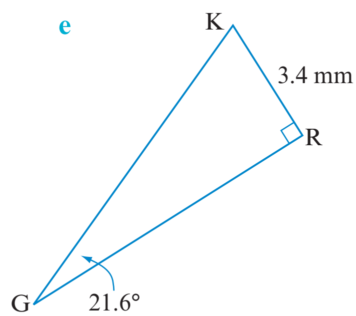

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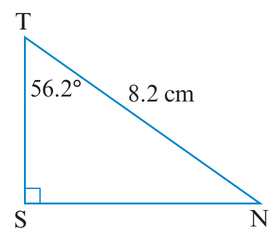
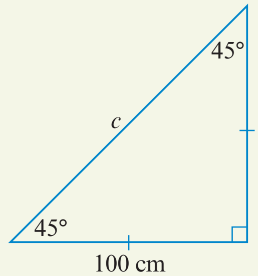


EXAMPLE 8B-2 Using an exact value of a trigonometric ratio to calculate an unknown side length

Use an exact value for the appropriate trigonometric ratio to calculate the length of the unknown side in this triangle. Give your answer as an exact value.



THINK

1 Form an equation using the relevant trigonometric ratio.

2 Replace sin 45° with its exact value and solve for the unknown value.

3 Write the answer as an exact value. Include the unit.

WRITE

$\sin 45° = \dfrac{100}{c}$

$\dfrac{1}{\sqrt{2}} = \dfrac{100}{c}$

$c \times \dfrac{1}{\sqrt{2}} = \dfrac{100}{c} \times c$

$\dfrac{c}{\sqrt{2}} = 100$

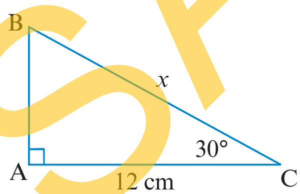
$c = 100 \times \sqrt{2}$

$c = 100\sqrt{2}$ cm

UNDERSTANDING AND FLUENCY

2 Use an exact value for the appropriate trigonometric ratio to calculate the length of the unknown side in each triangle. Give your answers as exact values.

a



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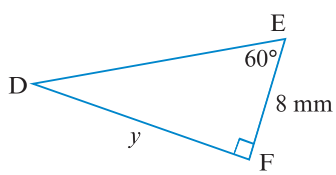

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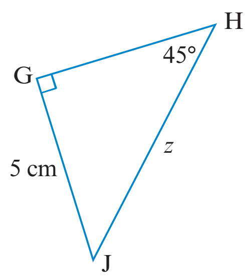

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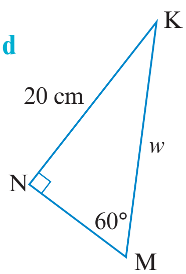

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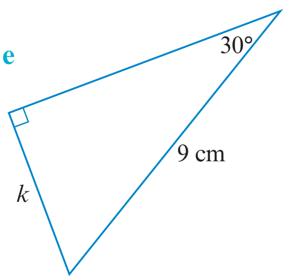

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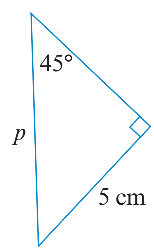


3 Calculate the approximate value, correct to one decimal place, for each exact length found in question **2**.

UNDERSTANDING AND FLUENCY

4 You may have noticed that the trigonometric values for some angles are related . Draw separate diagrams to explain each of these.

a Why does $\sin 30° = \cos 60°$ and $\cos 30° = \sin 60°$?

b Why does $\sin 45° = \cos 45°$?

c Why does $\tan 30° = \dfrac{1}{\tan 60°}$?

5 It is important when solving equations involving trigonometric ratios to take care when transposing. Each of these calculations contains an error. Explain where the error first occurs, then complete the solution correctly.

a
$$\begin{aligned}\cos 45° &= \frac{x}{12}\\ \frac{\sqrt{3}}{2} &= \frac{x}{12}\\ 12 \times \frac{\sqrt{3}}{2} &= 12 \times \frac{x}{12}\\ 6\sqrt{3} &= x\end{aligned}$$

b
$$\begin{aligned}\sin 30° &= \frac{8}{y}\\ \frac{1}{2} &= \frac{8}{y}\\ \frac{1}{2} \times 8 &= y\\ 4 &= y\end{aligned}$$

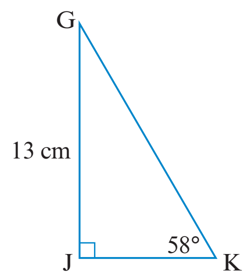
c
$$\begin{aligned}\tan 60° &= \frac{b}{7}\\ \frac{1}{\sqrt{3}} &= \frac{b}{7}\\ 7 \times \sqrt{3} &= b\\ 7\sqrt{3} &= b\end{aligned}$$

d
$$\begin{aligned}\tan 20° &= \frac{4}{z}\\ 4 \times \tan 20° &= \frac{1}{z}\\ \frac{1}{4 \tan 20°} &= z\end{aligned}$$

e
$$\begin{aligned}\cos 78° &= \frac{m}{12}\\ \frac{\cos 78°}{12} &= \frac{1}{m}\\ \frac{12}{\cos 78°} &= m\end{aligned}$$

f
$$\begin{aligned}\sin 15° &= \frac{a}{9}\\ \frac{1}{\sin 15°} &= \frac{9}{a}\\ a &= \frac{9}{\sin 15°}\end{aligned}$$

6 It is sometimes possible to simplify calculations by selecting a different angle for the trigonometric ratio.



a To calculate the length of JK you could say $\tan 58° = \dfrac{13}{\text{JK}}$. Solve this equation to find the length of JK.

b If you use the tangent of ∠JGK instead of ∠JKG, the side length JK would be in the numerator.

i What is the value of ∠JGK?

ii Use the tangent of this angle to calculate the length of JK.

iii Compare your answer with the one you obtained in part **a**.

c Comment on using the two different procedures for finding the length of the same line segment.

7 Since 1988, Australian currency has gradually been changed over from paper notes to polymer notes. The size has also changed. The paper \$5 note was 152 mm long, while the polymer one is 130 mm long. Interestingly, the diagonal of each of these notes forms the same angle of 26.6° with the longer side of each.

a Calculate the dimensions of the paper and the polymer \$5 notes. Give your answers to the nearest millimetre.

b Compare the dimensions of the two \$5 notes. What do you notice?

UNDERSTANDING AND FLUENCY

8 Kate is flying a kite on a 72-m string which makes an angle of 32° with the horizontal. Kate is 172 cm tall and is holding the end of the string at head height.

- a Draw a labelled diagram to display this information.
- b Calculate the vertical height of the kite above Kate.
- c How high is the kite above the ground?
- d What is the horizontal distance between Kate and the kite?

9 Shun is walking his dog on a leash. The leash is 2.5 m long, and his dog is 55 cm tall. His dog is pulling him along so that the taut leash makes an angle of 85° with the vertical.

- a Draw a diagram to display the information.
- b How far in front of Shun is his dog?
- c What is the vertical height of the end of the leash in Shun's hand above the ground?

10 Pythagoras' Theorem and the trigonometric ratios can both be used to calculate a side length in a right-angled triangle. Give an example of where:

- a Pythagoras' Theorem could be used, but trigonometry could not
- b trigonometry could be used, but Pythagoras' Theorem could not
- c either Pythagoras' Theorem or trigonometry could be used.

11 The front face of a brick has a diagonal length of 242 mm. This diagonal forms an angle of 18.3° with the longer side of the face. What are the dimensions of the face of the brick (to the nearest mm)?

PROBLEM SOLVING AND REASONING

12 Two bushwalkers start walking from their campsite. Roger walks due north, while Ben walks NNE. They both walk 25 km.

- a Display the information on a diagram.
- b How far north of his starting point is Ben when he completes his walk?
- c How far east of his starting point is Ben at this stage?
- d If Ben were to walk directly to meet Roger at his finishing point, how far would he have to walk?

13 Parliament House in Canberra occupies a site of 32 hectares. The building itself occupies only 15% of the site. It is 300 m long and 300 m wide, being one of the largest buildings in the southern hemisphere.

- a Draw a labelled diagram to show the floor plan of the building.
- b Use Pythagoras' Theorem to calculate the diagonal length of the building.
- c Use trigonometry to calculate the diagonal length of the building. (Hint: what is the size of the angle you can use?)
- d Comment on your answers to parts **b** and **c**.

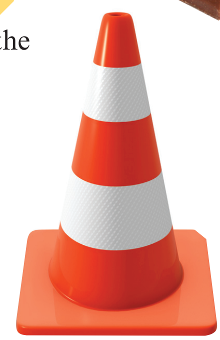
PROBLEM SOLVING AND REASONING

14 The Great Hall of Parliament House displays a tapestry based on a painting by the Australian artist Arthur Boyd. It measures 20 m wide and is claimed to be one of the largest tapestries in the world. The tapestry is floor to ceiling, with the angle between the diagonal and the floor being 24.2°.

- **a** Draw a diagram to display these facts.
- **b** How high is the ceiling in the Great Hall?
- **c** What is the perimeter of the tapestry?

15 A 440-g block of Toblerone chocolate is packaged in a box the shape of a triangular prism. The triangular faces of the box are equilateral in shape, with side lengths of 6 cm.

- **a** Calculate the height of the triangular face of the box using:
  - **i** Pythagoras' Theorem **ii** trigonometry.
- **b** The diagonal of the side rectangular face of the box is at 11.3° to the longer side of the face. What is the length of the chocolate box?



16 A traffic cone is 500 mm tall. (Assume that the sides meet at the top at a point.) The sides of the cone form an angle of 78.7° with the diameter of the base.

- **a** Calculate the length of the slant height of the cone.
- **b** What is the circumference of the cone at its base?

17 Sandy is standing 2 m in front of a picture hanging on a wall. The picture is 100 cm wide, and is hung so the bottom frame is 1.5 m from the floor. If she lowers her eyes 5.7°, she sees the bottom frame of the painting, while if she raises her eyes 16.7° she sees the top frame.

- **a** At what height are Sandy's eyes?
- **b** What are the dimensions of the painting?

CHALLENGE

18 A hexagon is a six-sided polygon. Consider a regular hexagon with side lengths of 12 cm. Calculate the distance between the parallel sides.

19 A pilot was flying a routine route from A, directly east to B, a distance of 3500 km. He didn't know his compass was not operating correctly, and he was actually travelling 10° south of his true direction. When he didn't arrive at his destination after travelling 3500 km, the pilot realised he was off course.

- **a** How much further does he need to travel to fly directly to his destination from this point? Give your answer to the nearest kilometre.
- **b** If the pilot had become aware of the error when he was half-way into the flight, and adjusted his direction from that point, how many kilometres would he have saved in the journey?

Reflect

Why can the answers be slightly different when trigonometry is used to solve a problem rather than Pythagoras' Theorem?

8C Finding angles using trigonometry

Start thinking!

One system of units used for the measurement of an angle is **degrees-minutes-seconds** (**DMS**).

1 degree = 60 minutes

1 minute = 60 seconds

An angle of 28 degrees 43 minutes and 17 seconds is written as 28°43′17″.

1 Your calculator has a button which allows you to convert from degrees into degrees, minutes and seconds. Convert these angles to degrees, minutes and seconds. Give your answer to the nearest second.

a 43.51° **b** 78.46° **c** 62.34° **d** 1.93°

2 Explain why each of these angles is not written correctly.

a 28°63′17″ **b** 49°13′67″ **c** 33°71′17″ **d** 15°23′60″

3 Round your answers in question **1** to the nearest minute.

The inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) or inverse tangent ($\tan^{-1}$) buttons on a calculator can be used to find the size of an angle from its trigonometric ratio.

4 Use your calculator to find the size of the angle θ for each given trigonometric ratio. Write your answers to the nearest:

i degree **ii** minute.

a $\sin\theta = 0.154$ **b** $\cos\theta = 0.926$ **c** $\tan\theta = 1.07$

d $\tan\theta = 0.864$ **e** $\sin\theta = 0.037$ **f** $\cos\theta = 0.829$

KEY IDEAS

- Angles can be written in degrees, minutes and seconds.
- There are 60 seconds in a minute and 60 minutes in a degree.
- When rounding to the nearest minute:
  - if there are less than 30 seconds, the number of minutes stays the same.
  - if there are 30 or more seconds, the number of minutes increases by 1.
- When rounding to the nearest degree:
  - if there are less than 30 minutes, the number of degrees stays the same.
  - if there are 30 or more minutes, the number of degrees increases by 1.
- A calculator can be used to obtain the size of the angle from its sine, cosine or tangent value. This is known as finding the inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) or inverse tangent ($\tan^{-1}$) of a value.

EXERCISE 8C Finding angles using trigonometry

UNDERSTANDING AND FLUENCY

1 Convert each angle to degrees and minutes. Give your answer to the nearest minute.

a $23.47°$ b $16.28°$ c $82.13°$ d $7.69°$
e $46.09°$ f $3.47°$ g $1.04°$ h $38.99°$

EXAMPLE 8C-1 Using a calculator to find angle sizes from trigonometric values

Calculate the size of θ if:

a $\tan\theta = 3.426$ b $\sin\theta = \frac{3}{13}$

Write your answers to the nearest:

i degree ii minute.

THINK	WRITE
a 1 Use the $\tan^{-1}$ key to find the angle size. Leave all decimal places on the calculator.	a $\tan\theta = 3.426$ $\theta = \tan^{-1}(3.426)$ $= 73.728\ldots°$
2 Round the answer to the nearest degree and minute.	i $\theta = 74°$ ii $\theta = 73°44'$
b 1 Use the $\sin^{-1}$ key to find the angle size. Leave all decimal places on the calculator.	b $\sin\theta = \frac{3}{13}$ $\theta = \sin^{-1}\left(\frac{3}{13}\right)$ $= 13.342\ldots°$
2 Round the answer to the nearest degree and minute.	i $\theta = 13°$ ii $\theta = 13°21'$

2 Use a calculator to find the size of θ for the given trigonometric ratios. Write your answers to the nearest:

i degree ii minute.

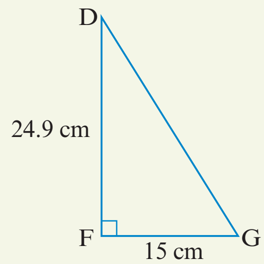
a $\sin\theta = 0.378$ b $\cos\theta = 0.845$ c $\tan\theta = 2.376$
d $\tan\theta = 0.097$ e $\sin\theta = 0.759$ f $\cos\theta = 0.238$
g $\tan\theta = 45.326$ h $\sin\theta = 0.019$ i $\cos\theta = 0.777$

3 Find the angle θ (to the nearest minute) for the given trigonometric values.

a $\cos\theta = \frac{4}{7}$ b $\tan\theta = \frac{15}{11}$ c $\sin\theta = \frac{4}{13}$
d $\tan\theta = \frac{57}{72}$ e $\sin\theta = \frac{29}{31}$ f $\cos\theta = \frac{3}{19}$
g $\tan\theta = \frac{100}{13}$ h $\cos\theta = \frac{44}{99}$ i $\tan\theta = \frac{1}{100}$

EXAMPLE 8C-2 Using trigonometric ratios to find unknown angles

Find the value of the two unknown angles in this triangle.
Give your answers to the nearest minute.

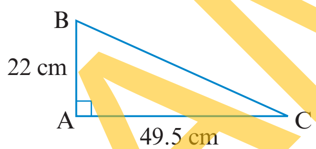


THINK	WRITE
1 The opposite and adjacent sides for ∠DGF are known. Use the tangent ratio.	$\tan \angle DGF = \frac{24.9}{15}$
2 Use the inverse tan. Leave all the decimal places showing on the calculator.	$\angle DGF = \tan^{-1}\left(\frac{24.9}{15}\right)$ $= 58.934\ldots°$
3 Convert to degrees and minutes, rounding to the nearest minute.	$\angle DGF = 58°56'$
4 Calculate the second unknown angle.	$\angle FDG = 180° - (90° + 58°56')$ $= 31°4'$

UNDERSTANDING AND FLUENCY

4 Find the value of the two unknown angles (to the nearest minute) in each triangle.

a



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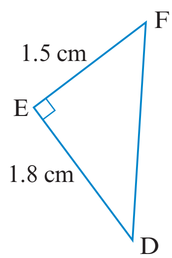

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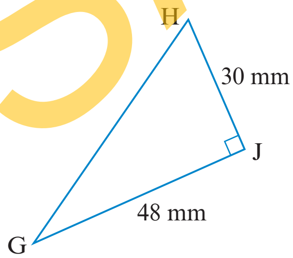

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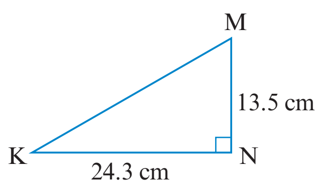

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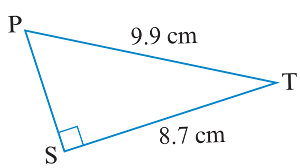

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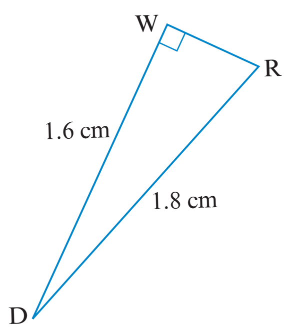


UNDERSTANDING AND FLUENCY

5 The Leaning Tower of Pisa is a freestanding bell tower, next to the cathedral in the Italian city of Pisa. The tower is 55.86 m tall on its low side and 56.7 m tall on its high side. The top of the tower is displaced 3.9 m horizontally from where it would be if it was standing vertically.

a The 'average' height of the tower above the ground represents the height of the centre of the top of the tower above the ground. Calculate this height.

b Draw a labelled sketch of the tower using this average height and its horizontal displacement. Mark the angle of 'lean' of the tower.

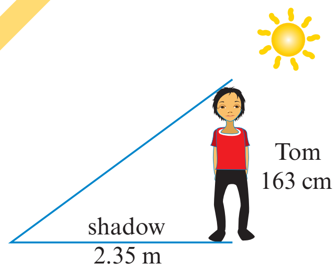
c Describe which trigonometric ratio you would use to calculate this angle.

d Calculate the angle of lean of the tower. Give your answer to the nearest:

i degree **ii** minute.

6 Tom and his brother Sam were standing in the sun measuring the length of each other's shadow. They each knew their height, so collated the measurements (see table).

	Height	Shadow length
Tom	163 cm	2.35 m
Sam	141 cm	2.04 m



a Tom drew a diagram to represent his shadow and height. Draw a diagram to display Sam's measurements.

b Calculate the base acute angle for each triangle. (Remember to take into account the difference in units.) Give your answers in degrees, correct to one decimal place. What does this base angle represent?

c **i** Explain why these two triangles must be similar in shape.

ii What is the linear scale factor in this case?

iii Draw the two triangles combined into one figure.

7 The sizes of Australian currency notes have changed over the years. Some of the new polymer notes are similar in shape to the previous paper ones, while others are a completely new shape. The table shows the dimensions of the notes.

Note description		\$5	\$10	\$20	\$50	\$100
Paper note	length	152 mm	155 mm	160 mm	165 mm	172 mm
	width	76 mm	76 mm	81 mm	82 mm	82.5 mm
Polymer note	length	130 mm	137 mm	144 mm	151 mm	158 mm
	width	65 mm	65 mm	65 mm	65 mm	65 mm

a Make a list of the obvious differences between the old notes and the new notes.

b One way to check whether the new note for a particular currency is similar in shape to its old equivalent is to find the size of the angle its diagonal makes with the length of the note.

i Find this angle for each of the old notes and new notes. Give your answers in degrees, correct to one decimal place.

ii Which notes have remained similar in shape?

PROBLEM SOLVING AND REASONING

8 Some road signs display the gradient of a road, particularly when approaching a hill or mountain climb. The gradient can be written in different forms, commonly as a ratio, a percentage or a fraction; for example, 1 in 10, 1:10, 10% and $\frac{1}{10}$ all represent the same gradient. Pat is a long-distance truck driver delivering goods between various locations. He generally plans his route to avoid steep hill climbs, if possible. His next delivery takes him via a mountainous stretch, where he has a choice of three possible routes.

Route 1: Over a horizontal distance of 12 km the rise is 6 km

Route 2: Over a horizontal distance of 18 km the rise is 7.5 km

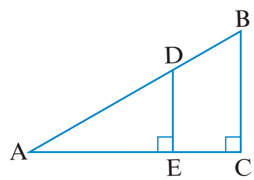
Route 3: Over a horizontal distance of 15 km the rise is 6.5 km

a Write the gradient of each route as a ratio and as a percentage.

b What is the angle of rise of each road over the distances given?

c Rank the routes in order of increasing steepness.

9 Consider this triangle. Within this figure, there are two right-angled triangles: ΔABC and ΔADE.



a Measure the relevant line segments (to the nearest millimetre), and complete the following, correct to two decimal places.

i $\sin \angle BAC = \frac{BC}{AB}$

$= \frac{\_\_ \text{ mm}}{\_\_ \text{ mm}}$

$= \_\_\_\_$

ii $\sin \angle DAE = \frac{DE}{AD}$

$= \frac{\_\_ \text{ mm}}{\_\_ \text{ mm}}$

$= \_\_\_\_$

iii Comment on your answers to parts **i** and **ii**.

iv The angle at A is the common angle in each of these calculations. Measure this angle. Use your calculator to find the sine of this angle. Comment on your answer.

b Repeat the calculations in part **a** for the cosine of angle A. Use the line segments relevant to the cosine ratio.

c Repeat the calculations in part **b** for the tangent of angle A. Use the line segments that give the tangent ratio.

d What do you conclude from your answers in parts **a**–**c**?

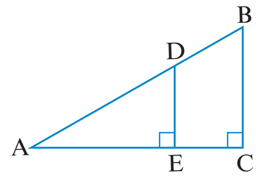
e **i** Explain why ΔABC and ΔADE are similar triangles.

ii Does similarity account for the fact that, for example, sin 30° is always 0.5, irrespective of the size of the right-angled triangle in which the 30° angle occurs? Explain your understanding.

iii Why does the ratio of a particular angle remain the same irrespective of the size of the triangle in which it occurs?

PROBLEM SOLVING AND REASONING

10 Investigate what happens to the value of the sine ratio as the angle increases from 0° to 90° in a right-angled triangle. Consider ΔABC in the diagram from question **9** shown on the right.



- **a** The sine of ∠BAC is the ratio of length of the opposite side to the length of the hypotenuse; that is, BC:AB. When the angle at A is 0°:
  - **i** what is the length of BC?
  - **ii** what does this mean for the value of the ratio BC:AB?
- **b** Now consider when the angle at A becomes almost 90°.
  - **i** What is the value of BC compared with AB?
  - **ii** What is the value of the ratio BC:AB?
- **c** Describe what happens to the value of the sine ratio as the angle increases from 0° to 90°.
- **d** Repeat parts **a**–**c** for the cosine ratio. Consider the relevant line segments for the cosine ratio.
- **e** Once more, repeat parts **a**–**c** for the tangent ratio, considering the relevant line segments for the tangent ratio.

11 From your investigations in question **10**, you can determine the minimum and maximum values for the three trigonometric ratios for angles in the range 0° to 90°. Copy and complete the following to summarise your findings.

The sine ratio has a minimum value of ____ and a maximum value of ____.
The cosine ratio has a minimum value of ____ and a maximum value of ____.
The tangent ratio has a minimum value of ____ and a maximum value of ____.

12 Check your answers to question **11** with a calculator. Provide some values to support your conclusions.

CHALLENGE

13 The A series of standard paper sizes used in Australia is based on the A0 size of paper having an area of 1 m^2. Its measurements are 841 mm by 1189 mm. The A1 size is formed from the A0 size by dividing the paper in half along its long side. This process continues for A2, A3, etc., with each size being derived from the previous one. You will be familiar with A4 size. When the measurements are halved, rounding down to the nearest millimetre generally occurs.

- **a** Draw a table to show the sizes of A0 to A5 paper.
- **b** Add a column to your table to show the size of the angle the diagonal makes with the longer side in each case. Give your values in degrees, correct to one decimal place.
- **c** It has been said that all the A series paper sizes are similar in shape. Comment on this statement.

Reflect

How can the value of the sine, cosine or tangent of an angle give an indication of the size of the angle?

8D Applications of trigonometry

Start thinking!

Trigonometry can be applied to many practical problems.

1 **a** Explain why you could not use Pythagoras' Theorem to calculate the height of this Qantas jet.

b Imagine you are standing 20 km from a point directly beneath the plane. You use a **clinometer** to measure the **angle of elevation** to the plane as 30°.

i Draw a labelled diagram to display this information.

ii Use a trigonometric ratio to calculate the height of this plane.

iii Calculate the straight-line distance between you and the plane.

c The plane approaches the airport at an altitude of 1000 m when it is 10 km horizontally from the airport. What is the **angle of depression** from the plane to the airport at that point?

2 Consider a Qantas jet on a flight to a destination 1000 km away with a **compass bearing** of N20°E.

a Draw a labelled diagram to display the distance travelled:

i north **ii** east.

b Use trigonometry to calculate these distances.

3 Sometimes a direction is given as a **true bearing**.

a What is a true bearing?

b Write the true bearings for flights to and from the destination in question **2**.

KEY IDEAS

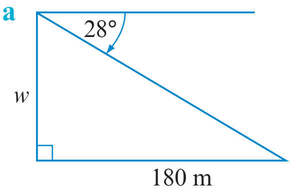
- The angle of elevation is the angle measured up from the horizontal to the line of vision.
- The angle of depression is the angle measured down from the horizontal to the line of vision.
- Compass bearings indicate direction from north or south towards east or west. For example, N30°W represents a bearing of 30° from north towards west.
- True bearings are angles measured from north in a clockwise direction. They are generally written using three digits, followed by a capital T. For example, 045°T represents 45° east of north, which is also north-east (NE).

EXERCISE 8D Applications of trigonometry

UNDERSTANDING AND FLUENCY

1 Use the angle of elevation or depression in each diagram to calculate the side length marked with the pronumeral. Give your answers correct to one decimal place.

a



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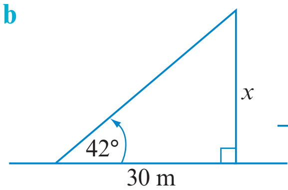

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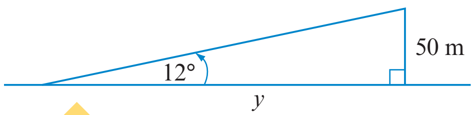

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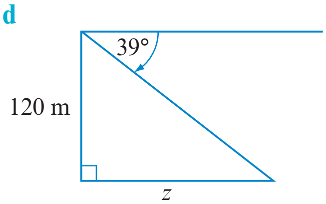

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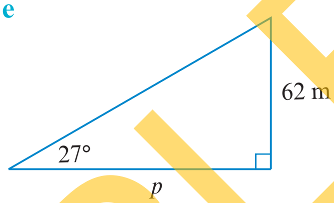

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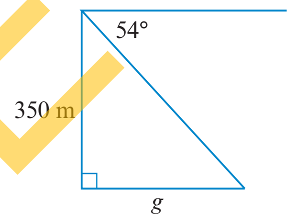
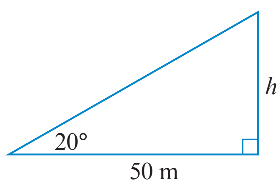


EXAMPLE 8D-1 Using trigonometry to calculate distances

The Cape Byron lighthouse is situated on a cliff at the most easterly point of the Australian mainland. From a point on the ground 50 m from the base of the lighthouse, the angle of elevation to the top of the lighthouse is 20°. Calculate the height of the lighthouse, to the nearest metre.

THINK	WRITE
1 Draw a diagram to represent the situation.	
2 Use the tangent ratio.	$\tan 20° = \dfrac{h}{50}$ $h = 50 \times \tan 20°$ ≈ 18.2
3 Answer the question, rounding to the nearest metre.	The lighthouse is 18 m high.

2 A lighthouse stands on a cliff above the sea. The angle of elevation to the top of the lighthouse from a point on level ground 100 m from its base is 14°. Find the height of the lighthouse, to the nearest metre.

3 The angle of depression from the light at the top of the lighthouse to a ship at sea 2.5 km from the base of the cliff is 3.3°. Find the height of the light above the sea.

EXAMPLE 8D-2 Using bearings with trigonometry

A ship at sea has travelled 1000 km on a bearing of N40°E since it left port.

a Draw a labelled diagram to display the distance travelled:

i east ii north.

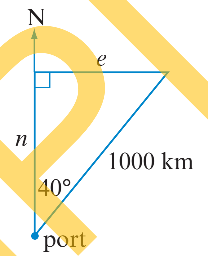
b Use trigonometry to calculate these distances to the nearest kilometre.

THINK

a Draw a diagram to represent the situation (e represents distance east, n represents distance north).

WRITE

a



THINK

b 1 Use the sine ratio to calculate the distance east.

WRITE

b $\sin 40° = \dfrac{e}{1000}$

$1000 \times \sin 40° = e$

$e \approx 642.8$

i The ship is 643 km east of the port.

THINK

2 Use the cosine ratio to calculate the distance north.

WRITE

$\cos 40° = \dfrac{n}{1000}$

$1000 \times \cos 40° = n$

$n \approx 766.0$

ii The ship is 766 km north of the port.

UNDERSTANDING AND FLUENCY

4 A ship at sea has travelled 250 km on a bearing of N30°E since it left port.

a Draw a labelled diagram to display the distance travelled:

i east ii north.

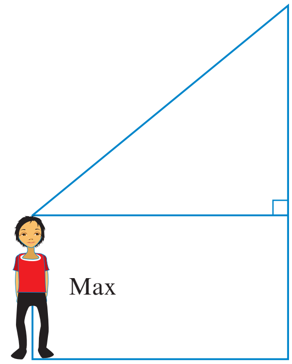
b Use trigonometry to calculate these distances to the nearest kilometre.

5 If the ship in question 4 turns around at this point and returns to its original port, write a description of the ship's path, giving the bearing and distances with respect to its turn-around position.

6 Write the true bearings for the ship's paths in questions 4 and 5.

UNDERSTANDING AND FLUENCY

7 Max threw a ball into the air. He viewed the ball at its highest point to be at an angle of elevation of 40° from him, and 3 m horizontally away from him. Max's eye level is 172 cm above ground.

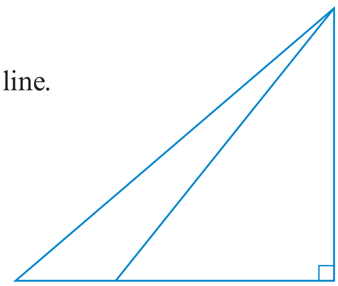


- **a** Copy and label this diagram with the known values.
- **b** What height does the ball reach above the level of Max's eyes?
- **c** What is the maximum height the ball reaches above the ground?
- **d** Why is it important to consider the height of a person's eyes in a situation like this?

8 Sedona is standing 20 m in front of a wall. From her eye level 168 cm above the ground, the angle of elevation to the top of the wall is 25°.

- **a** Draw a diagram to display the information.
- **b** Find the height of the wall.

9 Ann is standing 8 m in front of a tree. She observes the angle of elevation to the top of the tree to be 52°. Beth is standing further from the tree than Ann, but in the same line. She observes the angle of elevation to be 40°. (Ignore the heights of the two girls in this problem.)



- **a** Copy and label this diagram with the known values.
- **b** Find the height of the tree from Ann's observations.
- **c** Use the tree's height, together with Beth's angle of elevation, to calculate Beth's distance to the tree.
- **d** What is the distance between Ann and Beth?
- **e** Suppose Beth had been standing in the same line as Ann, but on the opposite side of the tree.
  - **i** Draw a diagram to show this situation.
  - **ii** What would be your answer to part **d** in this case?

10 A lookout tower is built on top of a cliff. At a point 50 m from the base of the cliff, the angle of elevation to the base of the tower is 58°, while the elevation to the top of the tower is 64°.

- **a** Draw a diagram to display this information.
- **b** Find the height of the tower.

PROBLEM SOLVING AND REASONING

11 Jake is riding his bike from home to his friend Luke's home. He rides 1.2 km directly north, then makes a right-angled turn left and continues for 3.2 km west to Luke's home.

- **a** Draw and label a sketch of Jake's route. Mark the north direction at each point.
- **b** Calculate the direct distance between the two homes.
- **c** Use trigonometry to find the bearing from Jake's home to Luke's. (Remember that bearings are measured from the north in a clockwise direction.)
- **d** What is the bearing from Luke's home to Jake's?
- **e** The bearing in part **d** is known as the **back-bearing** of the bearing in part **c** (and vice versa). What do you notice about the relationship between the two bearings?
- **f** Draw a scale drawing of Jake's route. From your drawing:
  - **i** find the distance between the two homes
  - **ii** use a protractor to measure the bearings in both directions.
- **g** Comment on any differences between your previous answers and those using your scale drawing.

12 Here's an activity you can do with a classmate.

- **a** Draw a scale drawing of the perimeter of your classroom. Mark your position anywhere within the room. Call this point X.
- **b** Measure the bearings from two adjacent vertices of your room plan to the point X.
- **c** Give this information to a classmate to locate your position. Note any questions your classmate asks regarding your instructions.
- **d** Describe how this information will define your position X.
- **e** Explore to see whether the position of X can still be located if bearings are made from any two vertices of your room plan.
- **f** Explain how your position could also be located by one bearing measurement and one distance measurement.

13 The members of a bushwalking group have been given instructions for their next hike.

- Start from a meeting place, labelled A, and walk 35 km at a bearing of 055°T to B.
- From B, walk 25 km at a bearing of 015°T to finish walk at C.

- **a** Draw a sketch of the walk. Try to make it roughly to scale as it can lead to incorrect calculations if the relative position of points is misleading.
- **b** Find how far north:
  - **i** B is from A **ii** C is from B **iii** C is from A.
- **c** Find how far east:
  - **i** B is from A **ii** C is from B **iii** C is from A.

PROBLEM SOLVING AND REASONING

d Consider the right-angled triangle in which AC is the hypotenuse. Calculate:
   i the distance from A to C
   ii the bearing from A to C
   iii the bearing from C to A.

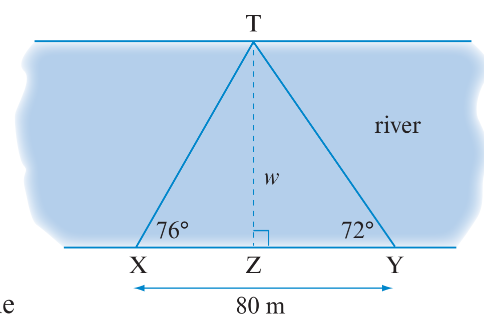
e Draw a scale drawing of the bushwalkers' route.
   i Measure distances and bearings to obtain answers for part **d**.
   ii Compare your answers with those for part **d**.

14 A new hike for bushwalkers took them along a different route.
From the starting point, walk 18 km at a bearing of 300°T.
Change direction and walk 23 km at a bearing of 200°T.
a Find the direct distance from the starting point to the end point.
b What is the bearing of the starting point from the end point?

15 Jan is standing 2 m in front of a painting on a wall. The bottom frame of the painting is 1.4 m from the floor, and Jan is 165 cm tall. The painting has a height of 70 cm.
a Draw a diagram to show this information.
b Find the angle of elevation from Jan's eyes to the top frame of the painting.
c What is the angle of depression from Jan's eyes to the bottom frame?
d If Jan stood 1 m further away from the painting, through what angle would she have to sweep her eyes to view from the top to the bottom frame?

CHALLENGE

16 Council surveyors have been set the task of measuring the width of the river along a straight stretch in preparation for building a bridge. They have located a tree on the opposite side of the river and taken angle measurements from two points X and Y, 80 m apart.
Calculate the width of the river. (Hint: let the distance XZ be represented by a pronumeral.)



Reflect

How are contour lines on maps related to angles of elevation?

8E Three-dimensional problems

Start thinking!

1 The Pyramid of Giza is the largest stone monument in the world. At one stage, it was also the tallest.
The base of the pyramid is a square with side lengths of 230 m.
It stands 140 m tall.
   - **a** Draw a labelled sketch of the pyramid.
   - **b** Calculate the diagonal length of the base.
   - **c** The apex of the pyramid lies directly over the midpoint of the base diagonal. Use the vertical right-angled triangle to calculate the angle the sloping edge makes with the base diagonal.
   - **d** A different right-angled triangle can be used to find the angle the sloping sides make with the base.
     - **i** Draw a labelled diagram to display its dimensions.
     - **ii** Calculate the angle the sloping side makes with the base.

2 Gustave Eiffel was the engineer in the company which designed and built the Eiffel Tower in Paris. The base of the tower is 120 m square, and its height is 320 m.
   - **a** Draw a sketch of the tower.
   - **b** Find the length of the diagonal of the base.
   - **c** The height of the tower from the midpoint of the base is 320 m.
     - **i** Draw a labelled diagram to find the distance from a corner of the base to the top of the tower.
     - **ii** What is this distance?
     - **iii** What is the angle this line makes with the base diagonal?

KEY IDEAS

- ▶ There can be many right-angled triangles in three-dimensional objects.
- ▶ Pythagoras' Theorem can be used to find an unknown side length of a right-angled triangle on the surface and within a 3D object.
- ▶ Trigonometry can be used to find an unknown angle or side length of a right-angled triangle on the surface and within a 3D object.

EXERCISE 8E Three-dimensional problems

1 Draw a sketch of each three-dimensional object. Highlight a right-angled triangle within each object.

a rectangular prism

b square-based pyramid

c triangular-based pyramid

d cylinder

EXAMPLE 8E-1 Finding lengths and angles within 3D objects

A box is in the shape of a cube with side lengths of 20 cm.

a Calculate the length of the diagonal of the base of the box.

b Find the length of the diagonal within the box.

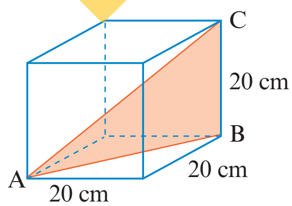
c Find the angle this internal diagonal makes with the base diagonal

THINK

a 1 Draw a labelled diagram of the box. Mark the relevant diagonals.

WRITE

a



THINK

2 Find the length of the base diagonal.

WRITE

length of base diagonal

$= \sqrt{20^2 + 20^2}$

$= \sqrt{800}$

≈ 28.3 cm

THINK

b Find the diagonal length within the box.

WRITE

b diagonal length within box

$= \sqrt{800 + 20^2}$

$= \sqrt{1200}$

≈ 34.6 cm

THINK

c 1 Define the right-angled triangle which contains the box's base diagonal and internal diagonal.

WRITE

c The base diagonal is AB.
The internal diagonal is AC.
The angle between these two is $\angle CAB$. It is within the right-angled triangle CAB.

THINK

2 Decide on the trigonometric ratio to use. Calculate the angle.

NOTE Any of the trigonometric ratios could be used here, as the lengths of all three sides of the triangle are known.

WRITE

$$\sin \angle CAB = \frac{20}{34.6}$$

$$\angle CAB = \sin^{-1}\left(\frac{20}{34.6}\right)$$

$$\approx 35°$$

The diagonal within the box makes an angle of 35° with the base diagonal.

UNDERSTANDING AND FLUENCY

2 A box is in the shape of a cube with side lengths of 15 cm.
  a Calculate the length of the diagonal of the base of the box.
  b Find the length of the diagonal within the box.
  c Find the angle this internal diagonal makes with the base diagonal.

3 A box has a base 20 cm by 15 cm and is 10 cm tall.
  a Provide working to show whether a 30 cm rod will fit in the box.
  b If a rod was a snug fit, what angle would it make with the base diagonal of the box?

4 A piece of cheese has the shape of a triangular prism. Its base is an isosceles triangle with side lengths of 10 cm, 10 cm and 4 cm, and its height is 5 cm.

  a Draw a labelled diagram of the triangular prism, showing all dimensions.
  b Draw a labelled diagram of:
    i the front view   ii the top view (or base view).
  c On your diagram of the front view, draw a diagonal.
    i Find the length of this diagonal.
    ii What angle does this diagonal make with the longer side?
    iii Without using trigonometry, explain how you could find the angle the diagonal makes with the shorter side.
  d Calculate:
    i the height of the base   ii the area of the base.
  e The cheese is sliced vertically through the base height.
    i Draw the shape of the face exposed. Label its dimensions.
    ii Find the area exposed.
    iii What is the diagonal length of the exposed face?

5 Jack was given a tent for Christmas. He wants to make a cover the same size to protect it when it rains. The front of the tent forms an equilateral triangle, with a height of 1.5 m.

  a Sketch the front of the tent. Label the height 1.5 m, and let the side lengths be x m. Highlight a right-angled triangle, and write the length of its base in terms of x.
  b Use Pythagoras' Theorem to find the value of x.
  c Jack has bought 3.5 m of material of the correct width to make the protective cover. Explain whether this is sufficient.

UNDERSTANDING AND FLUENCY

6 The rectangular perimeter of a football field is lined with trees. Ethan is standing 70 m in front of one of the trees. He notices a much taller tree 37° to his left. He measures the angle of elevation to the top of the taller tree as 13°.

- **a** Draw a sketch showing the relative position of the two trees described. Label the known values.
- **b** Find the distance along the ground from Ethan to the base of the taller tree.
- **c** Draw a sketch showing the elevation from Ethan to the tall tree. Label the known values.
- **d** Use this diagram to find the height of the tall tree.

7 Cooks sometimes find that if they leave their stirring spoon unattended, it falls into the saucepan.

This saucepan is in the shape of an cylinder open at one end, with a circumference of 60 cm and a height of 12 cm.

- **a** Find the diameter of the saucepan.
- **b** A stirring spoon is 20 cm in length. Explain whether it can fall into the saucepan.

8 An ice cream cone has a height of 10 cm. The circumference of the top circle of the cone is 15.5 cm.

- **a** Find the radius of the top circle.
- **b** Draw a labelled two-dimensional sketch of the outline of the cone. Define any right angles within the shape.
- **c** Find the slant length of the cone.

9 A Christmas tree in the shape of a cone is 3 m tall with a base diameter of 2 m.

Sandy wants to put streamers from the top to the base of the tree, along the outer surface of the tree. She has a roll of streamers 30 m long she can cut them up from. How many full length streamers is she able to attach?

10 A solid cube of side length 5 cm is sliced vertically through its base diagonal.

- **a** What is the shape of its exposed face? Draw a diagram with its dimensions labelled.
- **b** Compare the area of the exposed face with that of one of the faces of the cube.

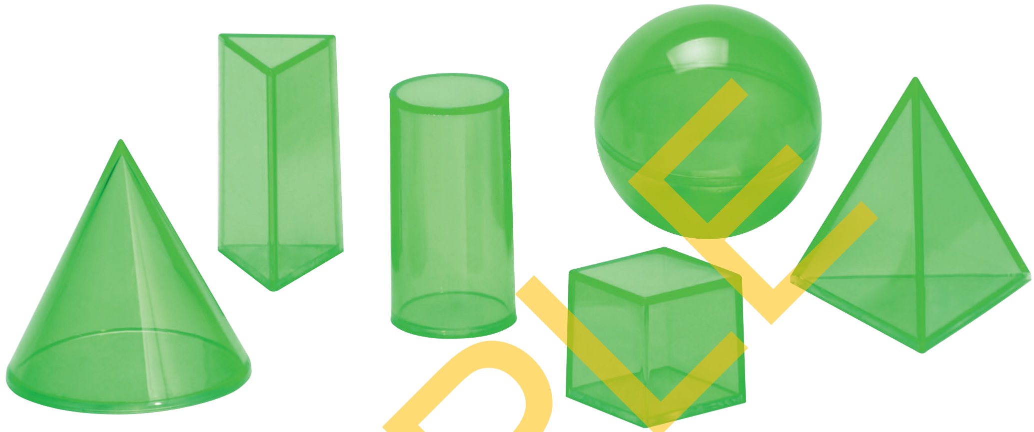
PROBLEM SOLVING AND REASONING

11 A hollow cube is made from a length of wire, bent to form the edges. The edges are each 5 cm long.

a Draw a diagram to display this cube.

b An ant starts crawling from a corner along the wire edges. What is the maximum distance the ant can crawl along the wire edges without going along the same edge twice? Explain your answer.

12 A **tetrahedron** is a three-dimensional object with four equilateral triangular faces.



a Identify the tetrahedron in this photo and draw a sketch. The length of each side is 10 cm and its height from base to the upper apex is 8.2 cm. Add these measurements to your sketch.

b i Draw a right-angled triangle displaying the height of the tetrahedron and the length of a side. Label the measurements.

ii Use your diagram to calculate the angle between the edge and a face in a tetrahedron.

c Draw the shape of a face. Mark in the perpendicular height, and label all known values.

i Find this perpendicular height.

ii Draw a right-angled triangle showing this perpendicular height of a face and the height of the tetrahedron. Use your diagram to calculate the angle between two faces of a tetrahedron.

13 A tower stands at point T on flat ground. From point X on the ground, the angle of elevation to the top of the tower (A) is 54°. At point Y on the ground, in the same straight line as XT, and 25 m beyond X, the angle of elevation to the top of the tower is 28°.

a Draw a labelled diagram to display the information. Label the height of the tower (AT) as h and the ground distance XT as d.

b In ΔAXT, use a trigonometric ratio to find an equation for h in terms of d.

c In ΔAYT, use a trigonometric ratio to find an equation for h in terms of d. (Note that the length of YT is $(25 + d)$ m.)

d Equate your two equations from parts **b** and **c**. Solve to find a value for d.

e Use this value for d in ΔAXT to find the height of the tower.

PROBLEM SOLVING AND REASONING

14 Three markers are placed on the ground at points A, B and C. The bearing of C from A is 030°T, and the bearing of C from B is 300°T. The bearing of B from A is 075°T, while the distance from A to C is 50 m.

- **a** Draw a diagram to show the position of the markers on the ground. Try to make your drawing as accurate as possible.
- **b** Find the distance from:
  - **i** B to C
  - **ii** B to A.
- **c** A flagpole is erected at marker A. The angle of elevation from C to the top of the flagpole is 35°. What is the angle of elevation from B to the top of the flagpole?

15 A rectangular prism has a base whose length is twice its width (w). The height of the prism is three times its width.

- **a** Write the length and height of the prism in terms of its width w.
- **b** Draw a labelled diagram to display the prism.
- **c** Find the length of the base diagonal in terms of w.
- **d** Calculate the length of the diagonal within the prism in terms of w.
- **e** Find the angle the internal diagonal of the prism makes with the diagonal of the base. Give your answer to the nearest degree.

CHALLENGE

16 The length of the diagonal within a cube is 6.2 cm. What is the side length of the cube?

17 Consider a cube of side length x cm.

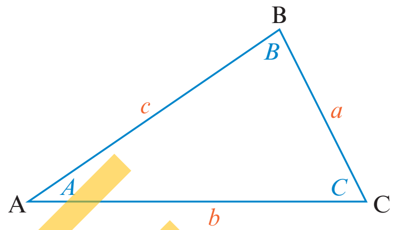
- **a** Draw a diagram of the cube.
- **b** An ant crawls from the top front left corner to the bottom back right corner along the outer surface.
  - **i** Draw a labelled net of the cube and label the ant's starting and finishing points. Mark the shortest path the ant could take.
  - **ii** Find the length of this shortest path. Give your answer in terms of the side length x.

Reflect

How can you decide whether to use Pythagoras' Theorem or trigonometry when determining a length in a right-angled triangle?

8F Sine and area rules

Start thinking!



Trigonometry can also be applied in non-right-angled triangles.

1 The convention is to label the side lengths with lowercase letters of the angles opposite these sides.

- **a** Copy this figure.
- **b** Draw a perpendicular from vertex B to base AC, meeting the base at D. Label the length of BD as h.
- **c**
  - **i** Use the sine ratio for angle A to write a relationship between h and c. Make h the subject of the formula.
  - **ii** Use the sine ratio for angle C to write a relationship between h and a. Make h the subject of the formula.
  - **iii** Equate your two answers and show how this can be written as $\dfrac{a}{\sin A} = \dfrac{a}{\sin C}$.

Similarly, relationships can be deduced using angle B. This relationship is called the **sine rule**, and can be written as: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ or alternatively as $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

These ratios are used in pairs to find an unknown angle or side length.

2 The area of a triangle is calculated as $\frac{1}{2} \times$ base $\times$ perpendicular height. If the triangle is not right-angled, and the perpendicular height is not given, it can be calculated using trigonometry.

- **a** Use the relationship you developed in question **1** part **ci** to show that the area of the triangle can be written as: area $= \frac{1}{2}bc \sin A$.
- **b** Using relationships between the other sides and angles, the formula for area can be written as:
  area $= \frac{1}{2}ab \sin C$ or area $= \frac{1}{2}ac \sin B$ or area $= \frac{1}{2}bc \sin A$
  To use this area formula, what necessary information do you need to know about a triangle?

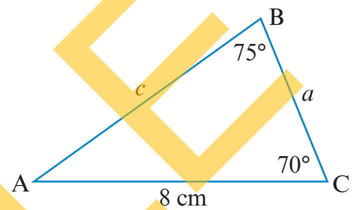
KEY IDEAS

- The sine rule can be written as $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ or alternatively as $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$. These ratios are used in pairs.
- The general formula for the area of a non-right-angled triangle can be written:
  area $= \frac{1}{2}ab \sin C$ or area $= \frac{1}{2}ac \sin B$ or area $= \frac{1}{2}bc \sin A$
- To find the area, two side lengths and the included angle must be known.
- If all the side lengths are known, but no angles are known, Heron's formula can be used to find the area of a triangular shape.
  area $= \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$

EXERCISE 8F Sine and area rules

EXAMPLE 8F-1 Using the sine rule

Find the unknown angle and side lengths (correct to one decimal place) for the triangle ABC, where $b = 8$ cm, $B = 75°$ and $C = 70°$.

THINK	WRITE
1 Draw a labelled triangle roughly to scale.	
2 Two angles are given, so the third angle can be calculated.	$A = 180° - (75° + 70°)$ $= 35°$
3 Use the sine rule to calculate the lengths of the other two sides. (Use the formula with the side length in the numerator.)	$\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ $\dfrac{a}{\sin 35°} = \dfrac{8}{\sin 75°}$ $a = \dfrac{8 \times \sin 35°}{\sin 75°}$ ≈ 4.8 cm $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ $\dfrac{8}{\sin 75°} = \dfrac{c}{\sin 70°}$ $c = \dfrac{8 \times \sin 70°}{\sin 75°}$ ≈ 7.8 cm

UNDERSTANDING AND FLUENCY

1 Use the sine rule to find the unknown value for each triangle ABC.

a If $A = 45°$, $B = 72°$ and $a = 10$ mm, calculate b.

b If $C = 65°$, $a = 12$ cm and $c = 13.4$ cm, find the size of angle A.

c $B = 37°$, $c = 8$ cm and $b = 15$ cm. Calculate A.

2 Find the unknown angles and side lengths for each triangle ABC using the given information.

a $a = 8.5$ cm, $B = 81°$ and $A = 37°$

b $c = 37.3$ mm, $b = 33.7$ mm, $C = 85°$

c $a = 22.5$ cm, $c = 28.3$ cm, $A = 49°$

d $b = 5.8$ cm, $B = 26°$, $A = 67°$

EXAMPLE 8F-2 Using the area rule

Find the area of triangle ABC shown in Example 8F-1 (correct to one decimal place).

THINK	WRITE
All the side lengths and angles are known, so any of the formulas can be used.	area $= \frac{1}{2}ab \sin C$ $= \frac{1}{2} \times 4.8 \times 8 \times \sin 70°$ $\approx 18.0\text{ cm}^2$

UNDERSTANDING AND FLUENCY

3 Find the area of each triangle ABC.

a $a = 9.5$ cm, $b = 7.6$ cm, $C = 28°$

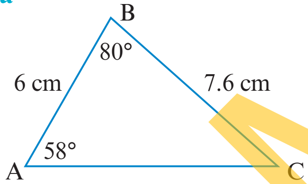
b $b = 2.8$ mm, $c = 8.9$ mm, $A = 87°$

c $a = 4.7$ cm, $c = 9.2$ cm, $B = 57°$

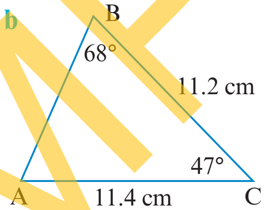
d $c = 12.5$ cm, $b = 9.2$ cm, $A = 43°$

4 Find the perimeter of each triangle.

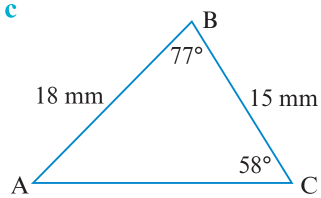
a



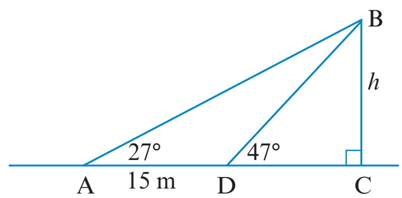
b



c



5 Chloe measures the angle of elevation to the top of a building as 27°. When she walks 15 m closer to the building, she measures the angle of elevation to be 47°.



a Find the size of:

  i ∠ADB

  ii ∠ABD.

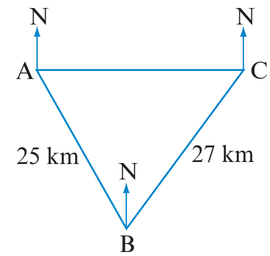
b Use the sine rule in ΔABD to calculate the length of BD, to one decimal place.

c Use the sine ratio in ΔBCD to find the height of the building.

d Use the cosine ratio in ΔBCD to find how far Chloe is from the building when she took her second reading.

e How far was she from the building initially?

6 A group of bushwalkers start from a point A, travels 25 km to a point B at a bearing of S30°E, then continues for 27 km to point C which lies due east of their starting point.



- **a** What is the value of ∠BAC?
- **b** Use the sine rule in ΔABC to find the size of ∠ACB to the nearest degree.
- **c** What is the size of ∠ABC?
- **d** Use the sine rule again to find the distance the bushwalkers would need to travel to return directly to their starting point. Give your answer to one decimal place.
- **e** Give the bearing:
  - **i** of C from B
  - **ii** of B from C
  - **iii** the bushwalkers would need to take to return directly to their starting point.
- **f** Calculate the area enclosed by the bushwalkers' hike.

7 The banks of a creek run directly east–west. From where Kim stands, the bearing to a tree on the opposite side of the creek is 032°T. If she moves 20 m due east on the bank of the creek, the bearing to the same tree is 310°T.

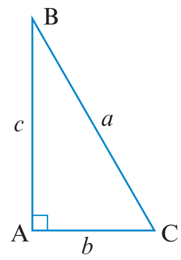
- **a** Draw a labelled diagram displaying as much information as possible.
- **b** Find the distance (in metres to one decimal place) of the tree from:
  - **i** Kim's first viewing point
  - **ii** Kim's second viewing point.
- **c** Calculate the width of the creek.

8 A lighthouse stands 50 m tall on top of a cliff. From a ship at sea, the angle of elevation to the base of the lighthouse is 3°, while the angle of elevation to the top of the lighthouse is 4.2°.

- **a** Draw a labelled diagram to display this information.
- **b** Use the sine rule to find the straight-line distance from the ship to:
  - **i** the base of the lighthouse
  - **ii** top of the lighthouse.
- **c** Calculate the distance in kilometres (to one decimal place) from the ship to the base of the cliff.
- **d** Find the height of the cliff, to the nearest metre.

UNDERSTANDING AND FLUENCY

9 Consider this right-angled triangle ABC.



The sine rule is quoted as $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

- **a** What is the value of $\sin A$?
- **b** Use the triangle to write expressions for $\sin B$ and for $\sin C$.
- **c** Substitute your expressions from parts **a** and **b** into the sine rule. Simplify.
- **d** Explain your answer. Does the sine rule hold for right-angled triangles?
- **e** What would have been your answer for part **c** if the triangle had been right-angled at B or C?

10 A give-way sign is the shape of an equilateral triangle, with side lengths of 40 cm. Calculate the area of the sign to the nearest square centimetre.

PROBLEM SOLVING AND REASONING

11 Refer to the diagram in question **9**. The area of the triangle can be calculated using the formula area = $\frac{1}{2}ab \sin C$.

- **a** Use the diagram to obtain an expression for $\sin C$.
- **b** Substitute this expression into the area formula and simplify.
- **c** Explain the resulting formula.

12 The formula you have used to find the area of a non-right-angled triangle relies on the fact that you know the length of two sides of the triangle, and the angle between these sides. If all the side lengths are known, but no angles are known, Heron's formula can be used to find the area of a triangular shape. (The proof is beyond this course, so is simply quoted here.)

$A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a + b + c)$, and a, b and c are the side lengths of the triangle. Note that s is the **semi-perimeter** of the triangle.

A triangular block of land has dimensions 200 m by 300 m by 400 m.

- **a** Draw a labelled diagram of the block. State values for a, b and c.
- **b** Find the value for s.
- **c** Substitute into Heron's formula. Simplify to find the area of the land to the nearest square metre.
- **d** If the block had been 300 m by 400 m by 500 m, it would be right-angled in shape.
  - **i** Explain why this is so.
  - **ii** Find its area using Heron's formula.
  - **iii** Calculate its area using the traditional area formula.
  - **iv** Compare your answers to parts **ii** and **iii**.

PROBLEM SOLVING AND REASONING

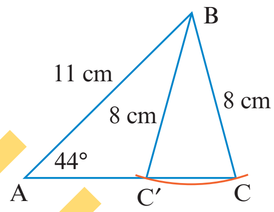
13 When using the sine rule, you have to make sure you are given enough information to define a specific triangle. If this is not the case, you may be able to draw two triangles from the given information, and therefore get two sets of answers. This is called the ambiguous case of the sine rule.

Consider a triangle ABC with $A = 44°$, $a = 8$ cm and $c = 11$ cm.

- **a** From this information, try to draw two different triangles. (Make sure they are roughly to scale.) Why is it possible to draw more than one triangle?

The two triangles can be drawn superimposed as shown here.

One solution belongs to triangle ABC (this is the one you would find with the usual sine rule calculations), while the other solution belongs to triangle ABC′. It is necessary to understand the relationship between angle C in the first triangle and angle C′ in the second triangle.



- **b** Consider ΔC′BC.
  - **i** What type of triangle is this?
  - **ii** What is the relationship between ∠BCC′ and ∠BC′C?
  - **iii** ∠BC′A is supplementary to ∠BC′C. Why is this so?
  - **iv** How would you find the size of ∠BC′A once you have found the size of ∠BCC′?
- **c** Use the sine rule within ΔABC to find:
  - **i** $\angle C$
  - **ii** $\angle B$
  - **iii** b.
- **d** In ΔABC′, using the sine rule where necessary, find:
  - **i** ∠BC′A
  - **ii** ∠ABC′
  - **iii** b.
- **e** Collate your two separate results to give the dimensions of the two triangles which satisfy the given information. Confirm that these results seem to be realistic values for the triangles drawn.

CHALLENGE

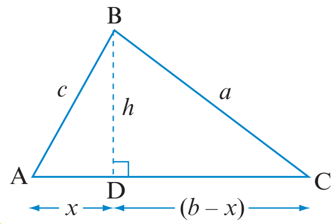
14 Question **4** part **c** is actually an ambiguous case where two distinct triangles can be drawn. Find a second set of solutions for this question.

Reflect

When would you use the area formula involving sine and when would you use Heron's formula to find the area of a triangle?

8G Cosine rule

Start thinking!



1 Consider triangle ABC, with the perpendicular from B dividing the base b into two sections of length x and $(b - x)$.

a Use ΔCDB and Pythagoras' Theorem to complete the relationship $a^2 = h^2 +$ __________.

b Use ΔADB and Pythagoras' Theorem to complete the relationship $h^2 =$ __________.

c Substitute the value of h^2 from part **b** into your equation in part **a**. Simplify the resulting equation and confirm that it can be written as $a^2 = b^2 + c^2 - 2bx$.

d Use ΔADB and the cosine ratio to write a relationship with x as the subject of the formula.

e Substitute your value for x into your equation in part **c**. Your resulting equation should be: $a^2 = b^2 + c^2 - 2bc \cos A$. This is known as the **cosine rule**.

f Similarly, expressions for b^2 and c^2 can be deduced.

g The cosine rule can be rearranged to find the size of an angle of a triangle. Confirm it can be written in the following forms.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

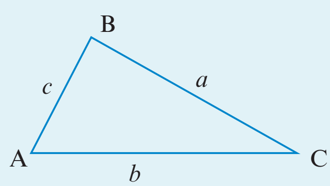
h Using the cosine rule, what information would you need to calculate:

i a side length?

ii an angle?

The cosine rule can be summarised as:
in any triangle ABC,
$a^2 = b^2 + c^2 - 2bc \cos A$
$b^2 = a^2 + c^2 - 2ac \cos B$
$c^2 = a^2 + b^2 - 2ab \cos C$

KEY IDEAS



- The cosine rule can be used to find the side length or angle of a triangle.
- For calculating a side length, the cosine rule can be written as:
  $a^2 = b^2 + c^2 - 2bc \cos A$ or $b^2 = a^2 + c^2 - 2ac \cos B$ or $c^2 = a^2 + b^2 - 2ab \cos C$

  Two side lengths and the size of the included angle must be known.
- For calculating an angle, the cosine rule can be written as

  $$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

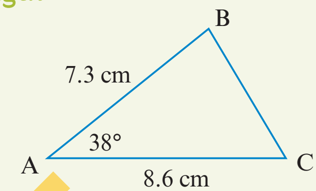
  The lengths of the three sides of the triangle must be known in this case.

EXERCISE 8G Cosine rule

EXAMPLE 8G-1 Using the cosine rule to find a side length

Consider this triangle.

Find the unknown side length, correct to one decimal place.



THINK

1 Identify the pronumerals.

2 Substitute into the cosine rule, leaving all the decimal places on the calculator.

3 Take the square root, and provide the answer.

WRITE

$b = 8.6$ cm, $c = 7.3$ cm, $A = 38°$

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$= 8.6^2 + 7.3^2 - 2 \times 8.6 \times 7.3 \times \cos 38°$$
$$= 28.307\ldots$$

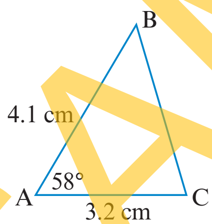
$$a = \sqrt{28.307\ldots}$$
$$\approx 5.3$$

The unknown side BC measures 5.3 cm.

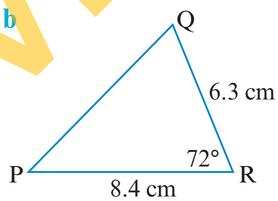
UNDERSTANDING AND FLUENCY

1 Find the unknown side length in each triangle, correct to one decimal place.

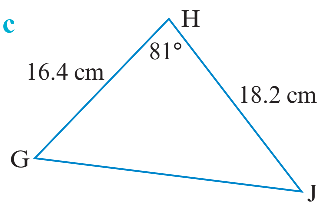
a



b



c



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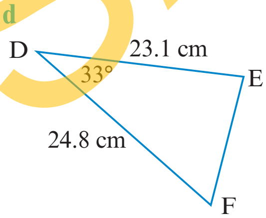

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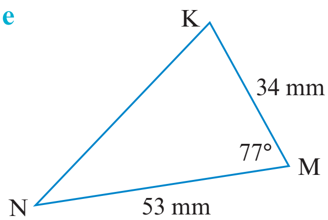

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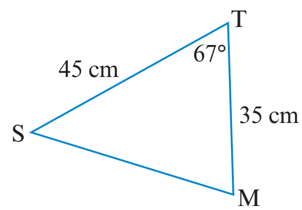
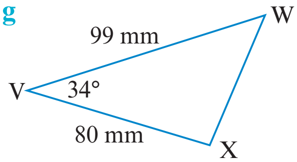
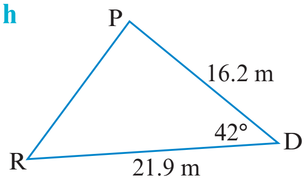


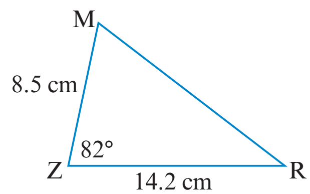
g



h



i



EXAMPLE 8G-2 Using the cosine rule to find an angle

Triangle ABC has side lengths of $a = 5.8$ cm, $b = 4$ cm and $c = 6.4$ cm.
Find the size of all angles in the triangle to the nearest degree.

THINK	WRITE
1 Use the angle form of the cosine rule for angle A. Substitute and simplify, leaving all decimal places on the calculator.	$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ $= \dfrac{4^2 + 6.4^2 - 5.8^2}{2 \times 4 \times 6.4}$ $= 0.4554\ldots$
2 Take the inverse cosine, rounding to the nearest degree.	$A = \cos^{-1}(0.4554\ldots)$ $\approx 63°$
3 Repeat steps 1 and 2 to calculate the size of angle B.	$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$ $= \dfrac{5.8^2 + 6.4^2 - 4^2}{2 \times 5.8 \times 6.4}$ $= 0.7893\ldots$ $B = \cos^{-1}(0.7893\ldots)$ $\approx 38°$
4 Calculate angle C.	$C = 180° - (63° + 38°)$ $= 79°$

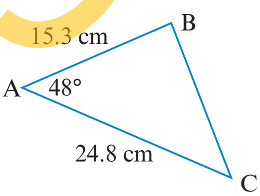
UNDERSTANDING AND FLUENCY

2 The following lengths are given for the sides in ΔABC. Find the size of all the angles.

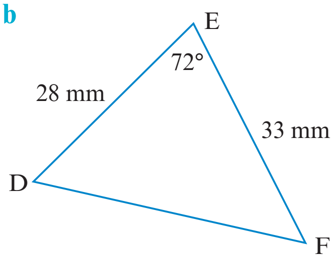
- **a** $a = 4.5$ cm, $b = 6$ cm and $c = 5$ cm
- **b** $a = 13$ mm, $b = 12$ mm and $c = 7$ mm
- **c** $a = 1.5$ m, $b = 2.3$ m and $c = 1.8$ m
- **d** $a = 16$ cm, $b = 19$ cm and $c = 12$ cm

3 Find the unknown side lengths and angles in each triangle.

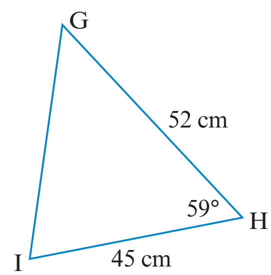
a



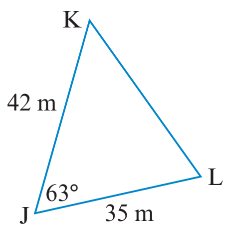
b



c



d



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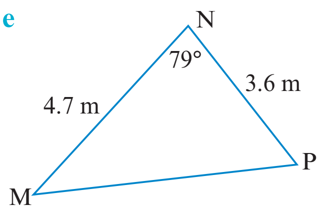

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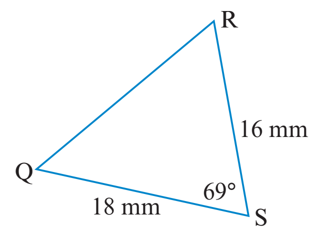
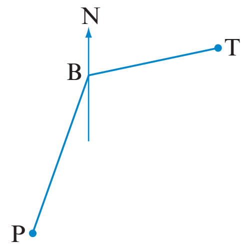


4 From a buoy at sea, Todd rows 2.5 km at a bearing of N80°E, while Pete rows 3.5 km at a bearing of S20°W.



- **a** Copy the diagram and label all known sides and angles.
- **b** What is the size of ∠TBP?
- **c** Join TP. This represents the distance between the two rowers. Use the cosine rule in ΔBTP to calculate this distance.
- **d** Use the cosine rule in ΔBTP to calculate the size of ∠BPT.
- **e** Use your answer from part **d** to find the bearing of Todd from Pete.
- **f** Calculate the size of ∠BTP.
- **g** Use your answer from part **f** to find the bearing of Pete from Todd.

5 A triangular garden bed has side lengths 5.3 m, 4.8 m and 4.5 m.

- **a** Draw a labelled sketch of the garden bed.
- **b** Identify the vertex with the largest angle. (Hint: the largest angle lies opposite the longest side.)
- **c** Use the cosine rule to find the size of this angle.
- **d** Explain how, using this angle, you can now calculate the area of the bed. What is its area?

6 A deep-sea diver is attached to two safety ropes from two boats on the surface of the ocean. The boats are 50 m apart. One rope is 110 m long, while the other is 100 m long.

- **a** Draw a labelled diagram to describe the situation.
- **b** Use the cosine rule to find the angles the two ropes make with the surface of the water.
- **c** Construct a right-angled triangle within your diagram which would enable you to find the maximum depth to which the diver could descend while attached to the two safety ropes. Explain a technique you could use to find this depth.
- **d** Calculate this maximum depth.

7 A give-way sign is in the shape of a triangle. Two adjacent sides measure 35 cm, and the angle between these sides measures 60°. Use the cosine rule to prove that the triangle is equilateral.

8 Investigate the cosine rule in right-angled triangles.

- **a** Draw a right-angled triangle ABC with the right angle at A.
- **b** Copy and complete the cosine rule: cos A = ____________________.
- **c** What is the value of cos A in your figure?
- **d** Substitute this value into your equation in part **b** and simplify. Describe the result.

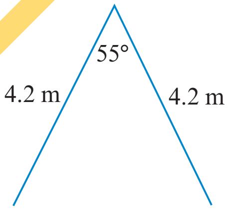
PROBLEM SOLVING AND REASONING

9 Emily is practising shooting hockey goals. The hockey goal is 3 m wide. She stands 5 m directly in front of one of the posts.

- **a** Draw a diagram to display the situation.
- **b** Use Pythagoras' Theorem to calculate her distance to the other goal post.
- **c** Use the cosine rule to find the angle within which she must shoot to score a goal.
- **d** Emily moves her position so she is 5 m from each of the goal posts. Explain whether she has a greater chance of scoring a goal from this position.

10 The end support pipes of a swing form a triangular shape with the ground.

The pipes are each 4.2 m long with an angle of 55° between them.



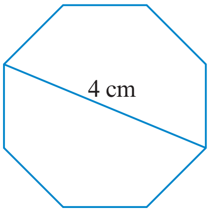
- **a** How far apart are the feet on the ground?
- **b** Find the vertical height of the top of the frame above the ground.

11 The frame for a bridge consists of a series of triangular shapes connected together as shown in the photo.

The sloping steel parts are 3.9 m long, while the horizontal steel parts are 2.8 m long.

- **a** Find the angle between the two sloping pieces of steel.
- **b** Draw a sketch of several sections of the bridge, showing the angles and lengths of the connecting pieces.
- **c** Find the height of the outside frame of the bridge if the horizontal beams are 10 cm wide.

12 The diagonal length of an octagon measures 4 cm.



- **a** Draw a sketch of the octagon, joining all the diagonals of the figure.
- **b** Consider one of the triangles within your figure.
  - **i** What is the size of the angle at the centre?
  - **ii** Find the sizes of the other two angles of the triangle.
- **c** Calculate the length of the outer edge of the octagon.

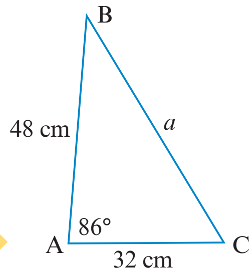
PROBLEM SOLVING AND REASONING

13 A light aircraft is set to fly on a course N78°E for a distance of 350 km. A strong easterly wind blows the plane off course by 6°.

a Draw a labelled diagram to show this situation.

b If the pilot flies the planned distance on this incorrect path, how far off course will he be at the end of his journey?

14 Sometimes there is a choice as to whether the cosine rule or the sine rule can be used to solve a problem. Consider this triangle.



a Explain why it is not possible to use the sine rule to calculate the value of a.

b Use the cosine rule to find the length of a.

c **i** Explain how you could now use the sine rule to find the sizes of angles B and C.

ii Calculate the values of these two angles.

d Angles B and C from part **c** could also have been calculated using the cosine rule. Find their values using this rule.

e Comment on your answers to parts **c ii** and **d**.

15 A cube has a side length of 25 cm.

a Draw a labelled sketch of the cube.

b Use the cosine rule to find the length of the diagonal of the base. Leave your answer in a simplified exact form.

c Use the cosine rule again to find the length of the diagonal of the cube. Leave your answer in a simplified exact form.

16 The angles of a triangle are in the ratio 3:4:5. The two sides that form the largest angle measure 3.5 cm and 4.3 cm.

a Find the size of all the angles.

b Calculate the length of the third side.

c Draw a labelled diagram to display the measurements of all the angles and sides.

CHALLENGE

17 A right-angled isosceles triangle is inscribed in a semi-circle of radius r.

a Draw a diagram to display this information.

b Find the lengths of the two equal sides in terms of r using:

i Pythagoras' Theorem

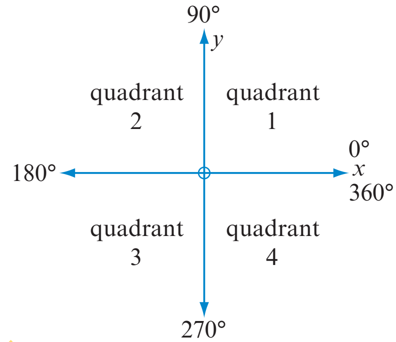
ii trigonometry.

c Comment on your two answers to part **b**.

Reflect

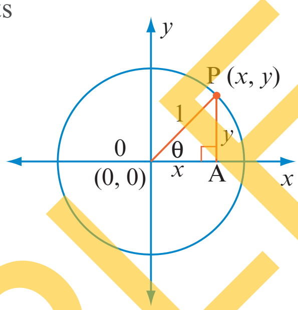
How can you tell when to use the sine rule and when to use the cosine rule in non-right-angled triangles?

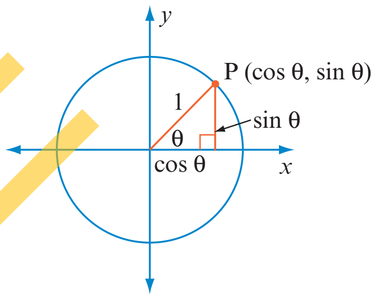
8H The unit circle and trigonometric graphs



Start thinking!

The **unit circle** is a circle drawn on the Cartesian plane with a radius of 1 unit and its centre at the origin. The quadrants on the Cartesian plane are labelled in an anticlockwise direction.





1 Imagine there is a point, P, with coordinates (x, y), on the circumference of the unit circle. As P moves around the circumference, the value of its coordinates changes. Consider its position as shown here, making an angle of θ with the positive direction of the x-axis.

- **a** Use a trigonometric ratio to find the values of x and y in terms of θ.
- **b** These labels can now be attached to the point P. So, P also has the coordinates $(\cos\theta, \sin\theta)$. Which trigonometric ratio represents:
  - **i** the horizontal direction?
  - **ii** the vertical direction?

2 The point P could be in any quadrant. The x- and y-values are positive or negative depending on the quadrant. Copy and complete this table.

	Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4
sign of x-coordinate $(\cos\theta)$				
sign of y-coordinate $(\sin\theta)$				
sign of $\tan\theta$ $\left(\frac{\sin\theta}{\cos\theta}\right)$				

An easy way to remember the sign of the trigonometric functions in the four quadrants is with a mnemonic – something like **A**ll **S**tations **T**o **C**anberra. In this case:

A: sin, cos and tan **all** positive in quadrant 1
S: **sin** positive in quadrant 2
T: **tan** positive in quadrant 3
C: **cos** positive in quadrant 4

KEY IDEAS

- The unit circle is a circle on the Cartesian plane with its centre at the origin and a radius of 1 unit.
- The x-coordinate of a point on the unit circle is represented by the cosine ratio.
- The y-coordinate of a point on the unit circle is represented by the sine ratio.
- Angles greater than 90° are generally represented by the angle made by the radius connecting the point with the x-axis.
- Sine, cosine and tangent values have different signs depending on the quadrant in which they lie. The mnemonic **A**ll **S**tations **T**o **C**anberra can be an aid to remembering these signs.
- A **periodic function** is one that repeats itself continuously in cycles.

EXERCISE 8H The unit circle and trigonometric graphs

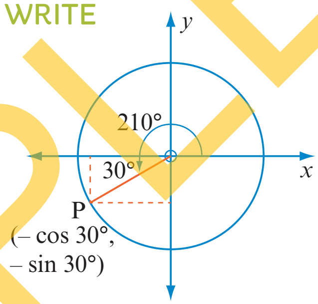
1 Identify the quadrant in which each angle lies.

a 345° **b** 103° **c** 204°

d 265° **e** 74° **f** 139°

EXAMPLE 8H-1 Using the unit circle to calculate exact trigonometric values for an angle greater than 90°

Use the unit circle and exact values for the trigonometric ratios to calculate the sin, cos and tan of 210°.

THINK	WRITE
1 Sketch the unit circle and mark the angle 210°. It is in quadrant 3 and is equivalent to (180° + 30°).	
2 Write values for sin 30° and cos 30° in exact form to calculate the exact value of tan 30°.	$\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{\sin 30°}{\cos 30°}$ $= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$ $= \frac{1}{\sqrt{3}}$
3 Apply the correct signs to the trigonometric ratios for quadrant 3.	In quadrant 3, sin and cos are both negative, while tan is positive. $\sin 210° = -\frac{1}{2}$ $\cos 210° = -\frac{\sqrt{3}}{2}$ $\tan 210° = \frac{1}{\sqrt{3}}$

2 Use the unit circle and exact values for the trigonometric ratios to calculate the sin, cos and tan of each angle.

a 240° **b** 300° **c** 120°

d 315° **e** 135° **f** 150°

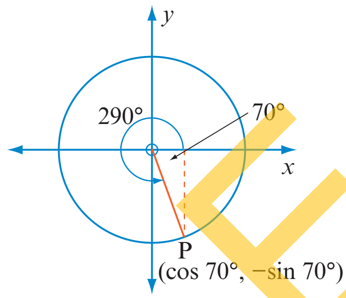
EXAMPLE 8H-2 Using the unit circle to calculate decimal trigonometric values for an angle greater than 90°

Use the unit circle to calculate the values of sin, cos and tan of 290°.
Give your answers correct to three decimal places.

THINK

1 Sketch the unit circle and mark the angle of 290°. It lies in quadrant 4 and is equivalent to (360° − 70°).

WRITE



In quadrant 4, sin is negative and cos is positive, so the coordinates of P are (cos 70°, −sin 70°).

THINK

2 Calculate values for sin 70° and cos 70°, correct to three decimal places. Use your calculator values to find the value of tan 70°.

WRITE

sin 70° = 0.940
cos 70° = 0.342

$$\tan 70° = \frac{0.940}{0.342} = 2.747$$

THINK

3 Apply the correct signs to the ratios.

NOTE These answers can be checked using a calculator. There may be a slight variation due to rounding.

WRITE

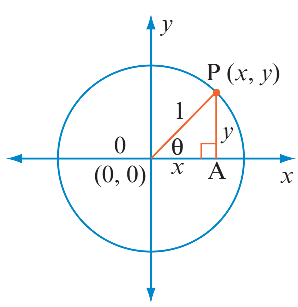
In quadrant 4, sin and tan are both negative, while cos is positive.
sin 290° = −0.940
cos 290° = 0.342
tan 290° = −2.747

UNDERSTANDING AND FLUENCY

3 Use the unit circle to calculate the values of sin, cos and tan of each angle. Give your answers correct to three decimal places.

a 87° b 285° c 202°
d 340° e 170° f 95°

4 Trace the point P around the unit circle in an anticlockwise direction from its starting point on the positive x-axis back to its starting point.



a You will notice that the value of the angle increases from 0° to 360°. If you consider the x value of the point as it moves from 0° to 90°, you notice that its value decreases. Describe what happens to the x value from:

i 90° to 180° ii 180° to 270° iii 270° to 360°.

b This change in x value is actually the change in the cosine of the angle as it sweeps from 0° to 360°. To obtain a clearer picture of this movement, you can create a table of cosine values for angles from 0° to 360°. Copy and complete this table with the aid of your calculator. Write your answers correct to one decimal place.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos θ													

c Plot these points on grid or graph paper, with the angle (θ) along the horizontal axis and the cosine value (cos θ) along the vertical axis. Join your points with a smooth curve. Describe the shape of the curve. What is the starting value of the curve?

d Sketch what would happen if you extend the angle values to 720°. This type of curve is known as a periodic function as it repeats itself continuously in cycles.

e The curve progresses through repeating **peaks** and **troughs**. The **period** of the curve is the distance between repeating peaks or troughs.

- **i** Note the value of cos θ at its first peak. How many degrees does the curve pass through before it reaches its next peak?
- **ii** What is the period of the **cosine graph**?

f The distance from the horizontal axis to a peak or a trough is called the amplitude of the graph. It is half the distance between the **maximum** and **minimum** points of the graph.

- **i** How far above the horizontal axis does the curve reach its maximum point?
- **ii** What is the distance from the horizontal axis to the curve's minimum point? (Ignore any negative signs.)
- **iii** What is the amplitude of the graph?

5 Consider the change in the y value as the angle sweeps from 0° to 360°. This actually represents the change in the sine value of the angle. Repeat the procedure in question **4** to graph the sine values for angles from 0° to 360° and then answer these questions.

a Describe what happens to the value of the sine of the angle from 0° to 360°.

b What is the starting sine value of the curve?

c Draw a sketch of the **sine graph**.

d Why would this be classed as a periodic function?

e What is the period of the graph?

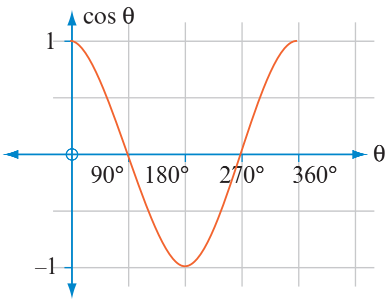
f How far above/below the horizontal axis does the curve reach its maximum/minimum?

g What is the amplitude of the graph?

6 The graphs of the sine function and the cosine function are quite similar. If you were presented with a cosine curve and a sine curve, and the graphs weren't labelled, how could you tell which was which?

UNDERSTANDING AND FLUENCY

7 Consider this graph of the cosine function.



a Use the graph to read the cosine value for each of these angles.

i 360° **ii** 90° **iii** 270° **iv** 180°

b Use the graph to read the approximate cosine value (correct to one decimal place) for each of these angles.

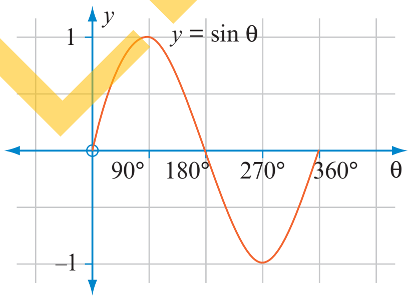
i 35° **ii** 100° **iii** 350°
iv 200° **v** 280° **vi** 140°

c Use the graph to find angles with these cosine values. Give your answers as approximate values, if necessary.

i 1 **ii** −1 **iii** 0
iv 0.4 **v** −0.6 **vi** 0.8

d Describe what you notice about your answers in part **c**.

8 Consider this graph of the sine function.



a Use the graph to read the sine value for each of these angles.

i 360° **ii** 90°
iii 270° **iv** 180°

b Use the graph to read the approximate sine value (correct to one decimal place) for each of these angles.

i 35° **ii** 100° **iii** 350°
iv 200° **v** 280° **vi** 140°

c Use the graph to find angles with these sine values. Give your answers as approximate values, if necessary.

i 1 **ii** −1 **iii** 0
iv 0.4 **v** −0.6 **vi** 0.8

d Describe what you notice about your answers in part **c**.

PROBLEM SOLVING AND REASONING

9 A grandfather clock has a pendulum which swings back and forth in a regular periodic motion. This has been one method of keeping time for hundreds of years, since the motion is so regular. The weight of the bob does not affect the period; however, the period is affected by the length of the pendulum.

a Explain why the motion of a pendulum is periodic.

b The equation used to find the period of a pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$, where T is the time (in seconds), l is the length of the pendulum (in metres) and g is the acceleration due to gravity (9.8 m/s^2). Find the period of a pendulum with lengths:

i 1 m **ii** 1.5 m **iii** 2 m

c Explain how the length of a pendulum affects its period.

10 Periodic functions occur in many everyday activities. Consider these sunrise and sunset times, recorded for the east coast of Australia on the first day of the month throughout a year. The times are recorded as Eastern Standard Time.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sunrise (am)	4.55	5.20	5.40	5.57	6.12	6.29	6.38	6.29	6.02	5.28	4.57	4.44
Sunset (pm)	6.46	6.42	6.20	5.46	5.17	5.01	5.04	5.19	5.34	5.47	6.05	6.28

a Plot the sunrise and sunset figures on separate graphs, placing the sunrise graph directly below the sunset graph. Write the months as the horizontal axis values and the time as the vertical axis values.

b Describe the shapes of the two curves.

c Are these two curves similar to those for sine and cosine functions? Explain any similarities or differences.

d Are the curves periodic? Explain why or why not.

e Because of the way you positioned these two graphs, the trend in daylight hours throughout the year is visually apparent. Explain how this is so.

f If you were not told that these were figures for the east coast of Australia, how could you reason that they must be figures for somewhere in the southern hemisphere?

11 The tangent ratio is also a periodic function.

a Complete a table of values for tan θ for θ values from 0° to 360°.

b Plot these points on a Cartesian plane and join them with a smooth curve.

c **i** Describe the shape of the graph.

ii It is different from the sine curve and the cosine curve. Why is this so?

iii At θ = 90°, a vertical asymptote occurs. Explain what you understand this to mean. Where does the next asymptote occur?

iv Explain why the graph is periodic. What is its period?

v What is the amplitude of the graph?

12 Repeat some of these questions using digital technology to confirm your answers.

Reflect

Music uses terms such as 'amplitude' and 'frequency'. What is the connection between these terms and trigonometric graphs?

8I Solving trigonometric equations

Start thinking!

Trigonometric functions are periodic, so trigonometric equations have infinitely many solutions.

1 Consider the graph for $y = \cos\theta$.

   a How many angles have a cosine value of 0.5 in the interval: **i** 0° to 360°? **ii** 0° to 720°

   b Use your calculator to find the number of answers.

A similar situation occurs when considering the graph of $y = \sin\theta$. In fact, any of the **trigonometric graphs** present the same property of multiple solutions. For this reason, in order to restrict the number of solutions, you generally solve the equation within a particular **domain** (the domain is the set of θ values).

Note that a calculator may not supply all the solutions within a quoted domain. You must look at the calculator answer, together with the domain, the period of the function and the sign of the trigonometric function in the various quadrants of the Cartesian plane. Considering all these options allows you to decide on the number of appropriate answers and their values.

2 Consider solving the equation $\sin\theta = \frac{1}{\sqrt{2}}$, $\theta \in [0°, 360°]$. (Note: [0°, 360°] means the interval of values from 0° to 360°. The initial and final values of the domain are shown within square brackets.)

   a What angle in the first quadrant has a sine value of $\frac{1}{\sqrt{2}}$?

   b What is the stated domain?

   c What is the period of the function and how many periods occur within this domain?

   d How many solutions for θ should you expect within this domain?

   e Is the sine value positive or negative in this equation?

   f What other quadrant/s have a sine value with this same sign?

   g Remembering that the angle is always measured from the positive or negative x-axis, what is another possible solution?

   h If the domain had been 0° to 720°, how many solutions would you expect? What are they?

3 Consider the equation $\cos\theta = -0.2$, $\theta \in [0°, 360°]$.

   a What angle in the first quadrant has a cosine of 0.2? Give your answer correct to the nearest degree.

   b The cosine value is negative in this equation. In which quadrants is cosine negative?

   c What are the solutions to this equation?

KEY IDEAS

- There are multiple solutions for trigonometric equations, as they relate to periodic functions.
- Sine, cosine and tangent values have different signs depending on the quadrant in which the angle lies. The mnemonic **A**ll **S**tations **T**o **C**anberra can be an aid to remembering these signs.
- A calculator supplies only one solution. Consider the quadrant in which the angle lies when searching for extra solutions.

EXERCISE 8I Solving trigonometric equations

UNDERSTANDING AND FLUENCY

1 Indicate whether the trigonometric value is positive or negative in each case.

a sin 100° **b** cos 200° **c** tan 140° **d** cos 295° **e** tan 260
f sin 340° **g** cos 94° **h** sin 130° **i** cos 190° **j** tan 350°

2 Write each angle in the form 180° − θ.

a 140° **b** 92° **c** 170° **d** 150° **e** 100°

3 Write each angle in the form 180° + θ.

a 200° **b** 260° **c** 190° **d** 220° **e** 255°

4 Write each angle in the form 360° − θ.

a 280° **b** 315° **c** 345° **d** 299° **e** 302°

EXAMPLE 8I-1 Solving a trigonometric equation within a given domain (exact value)

Solve the equation $\sin\theta = -\frac{1}{2}$, $\theta \in [0°, 360°]$.

THINK	WRITE
1 Write the equation.	$\sin\theta = -\frac{1}{2}$
2 First find the angle in the first quadrant that has a sine of $\frac{1}{2}$.	$\sin^{-1}\left(\frac{1}{2}\right) = 30°$
3 Identify the quadrants where sine is negative.	Sine is negative in quadrants 3 and 4.
4 Write possible values for θ.	So, θ = 180° + 30° = 210° or θ = 360° − 30° = 330°
5 Write the solution to the equation.	There are two solutions: θ = 210° or 330°

5 Solve each equation for the domain [0°, 360°].

a $\cos\theta = \frac{\sqrt{3}}{2}$ **b** $\sin\theta = -\frac{1}{\sqrt{2}}$

c $\cos\theta = -\frac{1}{2}$ **d** $\sin\theta = -\frac{\sqrt{3}}{2}$

e $\cos\theta = \frac{1}{\sqrt{2}}$ **f** $\sin\theta = \frac{\sqrt{3}}{2}$

EXAMPLE 8I-2 Solving a trigonometric equation within a given domain (decimal value)

Solve the equation $\cos \theta = 0.4$, $\theta \in [0°, 360°]$.

THINK	WRITE
1 Find the angle which has a cosine of 0.4.	$\cos \theta = 0.4$ $\cos^{-1}(0.4) = 66°$
2 Identify the quadrants where cosine is positive.	Cosine is positive in quadrants 1 and 4.
3 Write possible values for θ.	So, $\theta = 66°$ or $\theta = 360° - 66° = 294°$
4 Write the solution to the equation.	There are two solutions: $\theta = 66°$ or $294°$

6 Solve each equation for the domain $[0°, 360°]$.

a $\cos \theta = -0.4$ **b** $\sin \theta = -0.3$
c $\cos \theta = 0.9$ **d** $\sin \theta = 0.6$
e $\cos \theta = 0.6$ **f** $\sin \theta = -0.7$

7 Use digital technology to confirm your answers to questions **5** and **6**.

PROBLEM SOLVING AND REASONING

8 The exact values of sine and cosine for the angles 30°, 45° and 60°, together with their multiples, can be recorded on the unit circle.

a Draw a unit circle. Mark the coordinates of the points where the circle cuts the horizontal and vertical axes. Also, mark points on the circle representing angles of 30°, 45° and 60°, together with their multiples up to 360°.

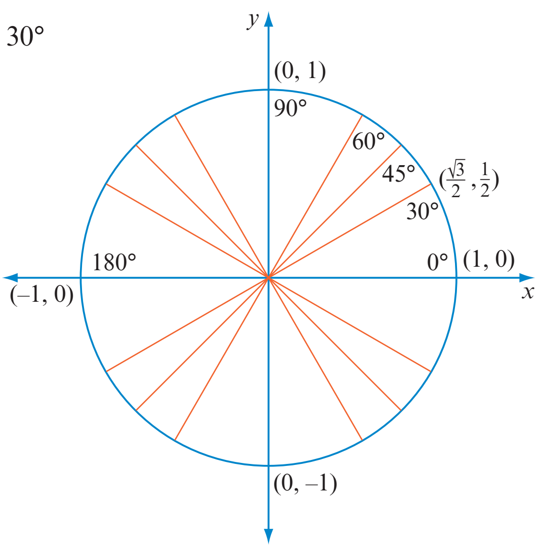
b Label each of the points shown on the unit circle with its exact cosine and sine value as coordinates.

For example, the point representing 30° would be labelled as $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Take care to use the correct signs.

c Give the exact coordinates of each of these angles on the unit circle.

i 225° **ii** 300
iii 150° **iv** 135°
v 315° **vi** 90°



9 Use your diagram from question 8 to find solutions to each equation for the domain [0°, 360°]. Remember to look for all the solutions.

a $\sin\theta = \frac{\sqrt{3}}{2}$ b $\cos\theta = \frac{\sqrt{3}}{2}$ c $\sin\theta = -\frac{\sqrt{3}}{2}$

d $\cos\theta = -\frac{\sqrt{3}}{2}$ e $\sin\theta = \frac{1}{2}$ f $\cos\theta = \frac{1}{\sqrt{2}}$

g $\sin\theta = -\frac{1}{\sqrt{2}}$ h $\cos\theta = -\frac{1}{2}$ i $\sin\theta = -1$

10 Consider the equation $\sin\theta + 2 = 3$, $\theta \in [0°, 360°]$.

a The equation simplifies to $\sin\theta = 1$. What angle has a sine value of 1? (Hint: refer to the unit circle.)

b Are there any other angles within the domain with a sine value of 1? If so, which one/s?

c What is the solution to the equation?

d Write a list of all the solutions for the domain [0°, 720°].

11 Consider the equation $\cos\theta + 1 = 0$. This can be rearranged as $\cos\theta = -1$.

a What solution/s can be found in the domain [0°, 360°]?

b What are the solutions in the domain [0°, 720°]?

12 It is possible that a trigonometric equation has no solution. Explain why each of these has no solution within any domain.

a $3 - \sin\theta = 1$ b $2\cos\theta = 3$ c $\frac{1}{4}\sin\theta = 1$

13 Indicate whether each equation has a solution within the domain [0°, 360°]. For those without a solution, explain why this is so. There is no need to solve the equation.

a $\cos\theta = \frac{1}{4}$ b $1 - \sin\theta = 1$ c $2\cos\theta = 2$

d $\frac{1}{2}\sin\theta = -\frac{1}{2}$ e $1 + \cos\theta = -1$ f $4\sin\theta = -\frac{1}{4}$

g $3 - \cos\theta = \frac{1}{3}$ h $\frac{1}{2}\cos\theta = -\frac{1}{4}$ i $\frac{1}{4} + \sin\theta = -1$

14 Take care when you consider the domain of a function. Don't always assume that a trigonometric function has two or more solutions. Consider the following functions within the domains indicated. Write the number of solutions for each. (Don't write the values of the solutions.)

a $\sin\theta = \frac{1}{2}$, $\theta \in [0°, 90°]$ b $\sin\theta = -\frac{1}{2}$, $\theta \in [0°, 90°]$

c $\cos\theta = -\frac{1}{2}$, $\theta \in [0°, 180°]$ d $\cos\theta = 0.3$, $\theta \in [0°, 180°]$

e $\sin\theta = -0.4$, $\theta \in [0°, 270°]$ f $\cos\theta = 0.8$, $\theta \in [0°, 270°]$

10A

PROBLEM SOLVING AND REASONING

15 Solve each equation for the domain indicated.

a $2 \sin \theta - 3 = \sin \theta - 2, \theta \in [0°, 180°]$

b $4 \cos \theta + 5 = 4 + 5 \cos \theta, \theta \in [0°, 180°]$

c $\sqrt{2} \cos \theta + 1 = 0, \theta \in [0°, 270°]$

d $2 \sin \theta + \sqrt{3} = 0, \theta \in [0°, 270°]$

16 Consider the situation where the domain does not start from 0°.

a What is the solution to $\sin \theta = 1$ within the domain 0° to 360°?

b If the domain started at the angle found in part **a** instead of starting at 0°, give the starting value of θ.

c There is only one solution from this point to 360°. How far would you have to extend the domain beyond 360° to get another solution?

d What is the smallest domain you could have if you wanted two solutions for the equation?

17 Give the smallest domain for each equation, given that two solutions are required.

a $\sin \theta = -1$ b $\sin \theta = 0$

c $\cos \theta = -1$ d $\cos \theta = 0$

18 Consult the cosine and sine values for angles on the unit circle.

a What is the smallest angle with a sine value of $\frac{1}{2}$?

b There is a larger angle with the same sine value. What is it?

c What would be the smallest domain for two solutions to the equation $\sin \theta = \frac{1}{2}$?

d What would be the smallest domain for four solutions to the equation?

19 Repeat question **18** for the equation $\cos \theta = \frac{1}{2}$.

20 Give the minimum domain for there to be two solutions for each equation.

a $\sin \theta = -1$ b $\cos \theta = 0$

c $\cos \theta = \frac{1}{2}$ d $\sin \theta = \frac{\sqrt{3}}{2}$

e $\sin \theta = \frac{1}{\sqrt{2}}$ f $\cos \theta = -\frac{\sqrt{3}}{2}$

21 Find the smallest domain for each equation to give:

i two solutions

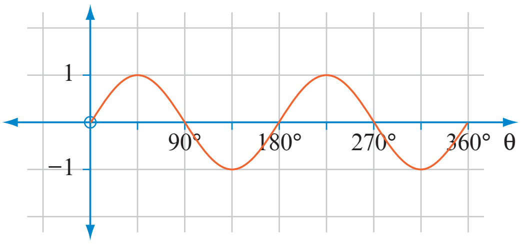
ii four solutions.

a $\sin \theta = -\frac{1}{2}$ b $\cos \theta = -\frac{1}{2}$

c $\sin \theta = \frac{1}{\sqrt{2}}$ d $\cos \theta = -\frac{1}{\sqrt{2}}$

CHALLENGE

22 Just as there are families of linear and quadratic functions, there are families of trigonometric functions.



a Explain why this graph is periodic.

b It is not the graph of $y = \sin\theta$ or $y = \cos\theta$. How can you tell?

c Does it more closely resemble a sine curve or a cosine curve? Explain.

d What is the period of the function?

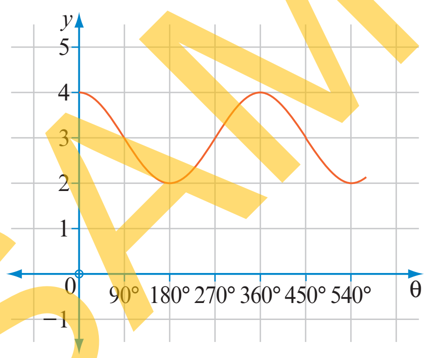
e How many solutions would you find for a particular positive value of the function (say 0.8) within the domain [0°, 360°]?

f How many solutions would you find for a particular negative value of the function (say −0.35) within the same domain?

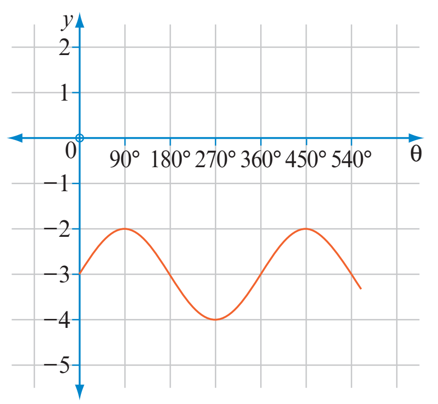
g How does this compare with the number of solutions within the same domain for $\cos\theta$ and $\sin\theta$?

23 These graphs are derived from graphs of $y = \sin\theta$ and $y = \cos\theta$. Study them carefully and indicate the trigonometric function each graph represents.

a



b



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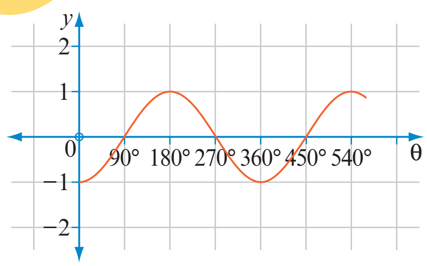

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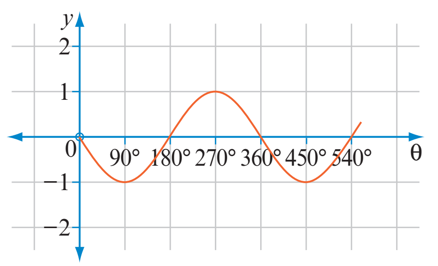


24 Use digital technology to check your answers to question **23**.

Reflect

How does symmetry in the unit circle help in solving trigonometric equations?

CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

Pythagoras' Theorem	degrees	back-bearing	peak
irrational number	minutes	sine rule	trough
Pythagorean triad	seconds	cosine rule	period of graph
trigonometry	angle of elevation	Heron's formula	amplitude
sine	angle of depression	unit circle	asymptote
cosine	compass bearings	trigonometric graphs	trigonometric equations
tangent	true bearings	periodic function	domain

MULTIPLE-CHOICE

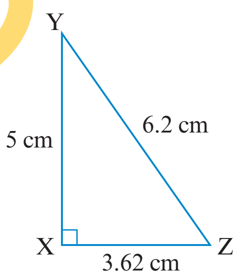
8A **1** Which of these describes a triangle with side lengths 4 cm, 5 cm and 6 cm?

A right-angled **B** obtuse-angled
C acute-angled **D** isosceles

8B **2** The ratio $\frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ of a right-angled triangle is known as which of the following?

A sine **B** cosine
C tangent **D** none of these

Questions 3 and 4 refer to this triangle.



8C **3** What is the size of ∠YZX, to the nearest degree?

A 54° **B** 55°
C 35° **D** 36°

8C **4** Which ratio represents sin ∠ZYX?

A $\frac{5}{6.2}$ **B** $\frac{3.62}{6.2}$ **C** $\frac{3.62}{5}$ **D** $\frac{5}{3.62}$

8D **5** What is the equivalent true bearing of N12°W?

A 348°T **B** 012°T
C 192°T **D** 168°T

8E 10A **6** Which value is closest to the diagonal length within a cube of side length 5 cm?

A 5 cm **B** 8 cm **C** 9 cm **D** 10 cm

8F 10A **7** What is angle A in $\frac{10}{\sin 80°} = \frac{9}{\sin A}$ closest to?

A 55° **B** 60° **C** 65° **D** 70°

8G 10A **8** The cosine rule is used to calculate the side length of a non-right-angled triangle ABC. If you know the value of angle B, what do you also need to know the value of?

A a and b **B** a and c
C c and b **D** angle A

8H 10A **9** A point P on the unit circle in quadrant 2 has what coordinates?

A $(-\cos\theta, \sin\theta)$ **B** $(\cos\theta, -\sin\theta)$
C $(-\cos\theta, -\sin\theta)$ **D** $(\cos\theta, \sin\theta)$

8I 10A **10** The graph of $y = \sin\theta$ has what period?

A 90° **B** 180° **C** 270° **D** 360°

8I 10A **11** What is the exact sine value for 30°?

A $\frac{1}{2}$ **B** $\frac{\sqrt{3}}{2}$ **C** $\frac{1}{\sqrt{3}}$ **D** $\frac{1}{\sqrt{2}}$

SHORT ANSWER

Calculate each angle correct to the nearest degree and each length correct to two decimal places, unless stated otherwise.

8A

1 An equilateral triangle has side lengths of 5 cm. Find the height of the triangle as:

- **a** an exact value
- **b** an approximate value, correct to one decimal place.

8B

2 A rectangle has a diagonal length of 15.8 cm. The diagonal forms an angle of 31.3° with the longer side. What are the dimensions of the rectangle?

8C

3 A map shows a stretch of road as a horizontal distance of 18.5 km. The road rises 6.2 km from start to finish.

- **a** Find the angle rise of the road, to the nearest minute.
- **b** Find the average gradient of the road as a:
  - **i** ratio
  - **ii** percentage.
- **c** What is the actual distance you would travel along the road?

8D

4 Two buildings stand 20 m apart. One is taller than the other. From the top of the taller building to the base of the shorter building, the angle of depression is 75°. From the top of the shorter building to the top of the taller building, the angle of elevation is 60°. What is the height of the shorter building?

Questions **5** and **6** refer to these three-dimensional objects.

- **a** cylinder with base circumference of 20 cm and height of 15 cm
- **b** box 10 cm long, 8 cm wide and 15 cm tall
- **c** cube of side length 25 cm

8E 10A

5 Find the length of the longest diagonal within each object.

8E 10A

6 Calculate the angle this diagonal makes with the height of the object.

8F 10A

7 Standing in a park, Anabel takes a bearing of 065°T to a tree. She walks 15 m due east and finds the bearing to the same tree to be 295°T.

- **a** Draw a labelled diagram to represent this situation.
- **b** Find the distance from Anabel, in both positions, to the tree.

8F 10A

8 In ΔABC, $a = 5$ cm, $b = 6.1$ cm and $A = 39°$.

- **a** Explain why two distinct triangles can be drawn with these measurements. Draw these two figures.
- **b** Find the dimensions of all the angles and sides of these two triangles.

8G 10A

9 A triangular garden bed in a park is 50 m by 40 m by 20 m.

- **a** Draw a sketch of the shape of the garden bed.
- **b** Find the area of the garden bed.
- **c** Find the size of the angle at each vertex of the triangular shape.
- **d** What is the shortest distance from the vertex opposite the longest side to the longest side?

8H 10A

10 Show how the unit circle can be used to find each value.

- **a** sin 300° **b** cos 225° **c** sin 135°
- **d** cos 315° **e** sin 240° **f** cos 330°

8I 10A

11 Solve each equation for the domain [0°, 360°].

- **a** $\cos\theta = -1$ **b** $\sin\theta = 0$
- **c** $\cos\theta = \frac{\sqrt{3}}{2}$ **d** $\sin\theta = -\frac{1}{2}$
- **e** $\cos\theta = -\frac{1}{\sqrt{2}}$ **f** $\sin\theta = -\frac{\sqrt{3}}{2}$

MIXED PRACTICE

Where appropriate, calculate each angle correct to the nearest degree and each length correct to one decimal place, unless stated otherwise.

1 A square has a side length of 3.2 cm. What is the length of its diagonal, correct to the nearest centimetre?

A 3 cm **B** 4 cm **C** 5 cm **D** 6 cm

2 What is the exact value of sin 60°?

A $\frac{1}{2}$ **B** $\frac{\sqrt{3}}{2}$ **C** $\frac{1}{\sqrt{3}}$ **D** $\frac{1}{\sqrt{2}}$

3 What is the angle between the diagonal and the side of a square?

4 A square has a diagonal length of $4\sqrt{2}$ cm. What is its side length?

5 From a point 5 km horizontally from the point directly beneath it, the angle of elevation to a flying plane is 9°. At what height is the plane flying?

6 What is the value of cos 60° the same as?

A cos 30° **B** sin 30° **C** sin 60° **D** cos 45°

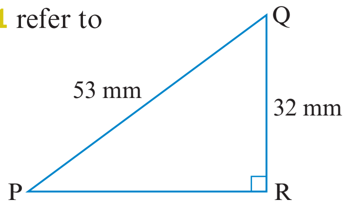
7 What true bearing is equivalent to the compass direction N52°W?

8 The hypotenuse of a right-angled triangle measures 8.4 cm. One of the other sides measures 6.3 cm. What is the length of the third side, to the nearest centmetre?

A 5 cm **B** 6 cm **C** 7 cm **D** 8 cm

9 Calculate the value of $\frac{5}{\cos 14.8°}$.

Questions **10** and **11** refer to this triangle.



10 Find the size of ∠PQR to the nearest minute.

11 Write an expression for the sine of ∠QPR.

12 What is the value of $\sin^{-1}\left(\frac{7}{9}\right)$, to the nearest minute?

A 51°3′ **B** 51°4′ **C** 51°5′ **D** 51°6′

10A **13** Write the cosine rule to find the value of an angle *A* in triangle ABC.

10A **14** What is the period of the graph of $y = \cos\theta$?

A 120° **B** 180° **C** 360° **D** 540°

10A **15** In which quadrant of the Cartesian plane is the sine value negative and the cosine value positive?

A quadrant 1 **B** quadrant 2
C quadrant 3 **D** quadrant 4

10A **16** Write the formula to find the length of a diagonal within a rectangular prism with dimensions *x* cm × *y* cm × *z* cm.

10A **17** How many solutions would the equation cos θ = −0.5 have within the domain [0°, 450]?

A 1 **B** 2 **C** 3 **D** 4

10A **18** Which angle in the first quadrant has an exact tangent value of $\sqrt{3}$?

A 30° **B** 45° **C** 60° **D** 90°

10A **19** In ΔABC, $c = 12$ cm, $A = 60°$ and $C = 50°$. Use the sine rule to calculate the value of a.

10A **20** Use Heron's formula to find the area of a triangle with side lengths 6 cm, 7 cm and 10 cm.

10A **21** Compare the graphs of $y = \cos\theta$ and $y = \sin\theta$.

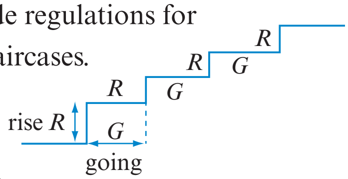
10A **22** Why is there an asymptote at 90° in the graph of $y = \tan\theta$?

10A **23** What is the amplitude of the graph of $y = \sin\theta$?

A 1 **B** 1.5 **C** 2 **D** 3

ANALYSIS

1 There are building code regulations for the construction of staircases. For safety purposes, the tread cannot be too narrow, nor the staircase too steep. There are particular terms for staircases, as shown in this diagram.



The rise (R) must have a minimum height of 115 mm and a maximum height of 190 mm.
The going (G) must have a length within the range 250–355 mm.
The regulation also states that $2R + G$ must lie within the range 550–700 mm.

a Consider a staircase with the minimum rise of 115 mm and the minimum going of 250 mm.

i Explain whether this staircase would fall within the building regulation guidelines.

ii What is the slope of this staircase? Give your answer to the nearest degree.

b Repeat part **a** with the maximum values for the rise and going.

c If the going is its maximum value of 355, and the value of $2R + G$ falls within the required limits:

i find a range of acceptable values for the rise

ii calculate the range of acceptable slope values.

d Draw a set of five stairs with a going of 300 mm and a rise of 150 mm.

i What is the horizontal distance of the stairs?

ii Find the vertical height of the stairs.

iii Calculate the direct distance between these two points.

iv What is the slope of the staircase?

e What do you consider would be ideal figures for the going and rise of a comfortable staircase? Draw a sketch of your design, show that it falls within the regulation guidelines, and give its slope.

f There are different building code regulations for wheelchair ramps. The slope of the ramp is of particular importance, as a wheelchair may tip backwards or run away if it is too steep. The code specifies that an internal ramp should be no steeper than 1 to 12, while an external ramp should have a gentler slope of 1 to 20.

i Draw internal and external ramps 20 m long with these slopes.

ii For each one, find the going and the rise. Mark these measurements on your diagrams.

10A **2** Emma rides her bicycle at a constant speed with a tag attached to a spoke at the rim of the wheel. Tom records the height of the tag with respect to its starting position with a video. Here is his record.

Time (s)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Height (cm)	20	30	20	0	−20	−30	−20	0	20	30

a Graph the data, showing time on the x-axis.

b Suggest whether the graph models a sine curve or a cosine curve.

c How long does it take for the wheel to rotate once?

d What was the starting position of the tag on the wheel? Explain how you deduced this.

e What is the radius of the wheel?

f Calculate the circumference of the wheel.

g At what speed does Emma ride? Give your answer in metres per second.

h Draw a diagram to show the position of the tag after 10 seconds of riding at this constant speed. Explain how you arrived at this answer.

CONNECT

How high is that?

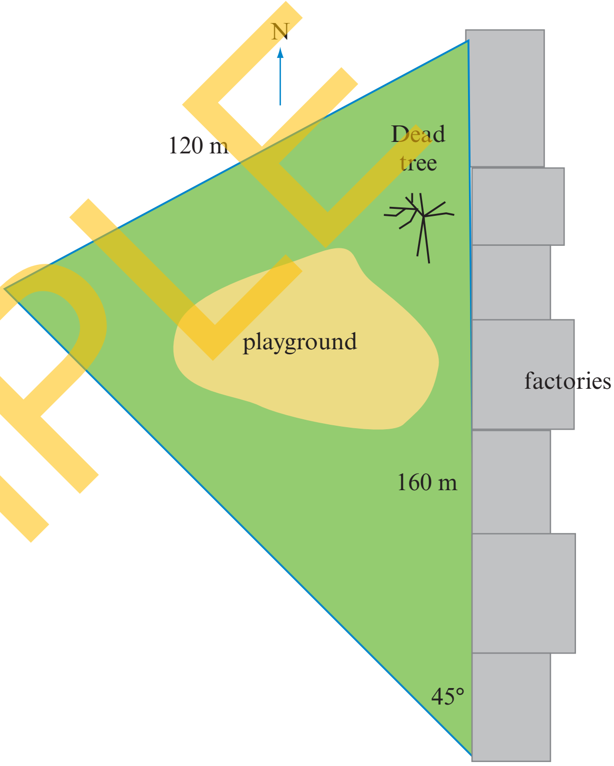
Joe is a landscape architect who works for the local council. His latest project is to design a recreation park on a disused block of land next to a block of factories. The land is triangular in shape.

A dead tree must be removed from the block before the landscaping can commence.

There's to be a playground area for the children with swings and gym equipment. The land must be level in this area.

Joe has also been asked to include a bed of sunflowers. Since these flowers follow the path of the sun in its daily journey across the sky, they must have full sun throughout the day for as much time as possible. This means that they can't be in the shade of the line of factories for any length of time.

Joe certainly has his work cut out for him! He starts by initially drawing a plan of the area, and the proposed improvements.



Your task

For your investigation of a landscape architect's work, complete these steps.

1 Research recreation sites and draw a proposal for the development of the block.

2 Explain how you catered for the constraints of:

- a cutting down the dead tree
- b levelling the playground
- c positioning the swings and gym equipment
- d positioning the sunflower garden.

Extension: The size of the land

10A 3 Use the measurements on Joe's plan to determine for the block:

- a the angles of the vertices
- b the lengths of all the sides
- c the perimeter
- d the area.

10A 4 Add this detail to your proposed plan.

Include all necessary diagrams and working to justify your answers.

You may like to present your findings as a report. Your report could include:

- an information booklet
- a PowerPoint presentation
- an advertising brochure
- other (check with your teacher).